

# Liquid Welfare Guarantees for No-Regret Learning in Sequential Budgeted Auctions

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## Abstract

We study the liquid welfare in sequential first-price auctions with budget-limited buyers. We focus on first-price auctions, which are increasingly commonly used in many settings, and consider liquid welfare, a natural and well-studied generalization of social welfare for the case of budget-constrained buyers. We use a behavioral model for the buyers, assuming a learning style guarantee: the resulting utility of each buyer is within a  $\gamma$  factor (where  $\gamma \geq 1$ ) of the utility achievable by shading her value with the same factor at each iteration. Under this assumption, we show a  $\gamma + 1/2 + O(1/\gamma)$  price of anarchy for liquid welfare assuming buyers have additive valuations. This positive result is in stark contrast to sequential second-price auctions, where even with  $\gamma = 1$ , the resulting liquid welfare can be arbitrarily smaller than the maximum liquid welfare, even though the latter can be achieved by a constant shading factor. We prove a lower bound of  $\gamma$  on the liquid welfare loss under the above assumption in first-price auctions, making our bound asymptotically tight. For the case when  $\gamma = 1$  our theorem implies a price of anarchy upper bound that is about 2.41; we show a lower bound of 2 for that case.

We also give a learning algorithm that the players can use to achieve the guarantee needed for our liquid welfare result. Our algorithm achieves utility within a  $\gamma = O(1)$  factor of the optimal utility even when a buyer’s values and the bids of the other buyers are chosen adversarially, assuming the buyer’s budget grows linearly with time. The competitiveness guarantee of the learning algorithm deteriorates somewhat as the budget grows slower than linearly with time.

Finally, we extend our liquid welfare results for the case where buyers have submodular valuations over the set of items they win across iterations with a slightly worse price of anarchy bound of  $\gamma + 1 + O(1/\gamma)$  compared to the guarantee for the additive case.

## 1 Introduction

In 2022, over 90% of all digital display ad dollars were transacted programmatically and accounted for over \$123 billion in the United States [Yue22]. This is usually done through auction platforms where the advertisers sequentially submit bids to place their ads at slots, as the latter become available. This is done in many platforms such as Google’s DoubleClick, more centralized platforms such as Facebook Exchange and Twitter Ads, as well as sponsored search platforms such as Google’s AdWords and Microsoft’s Bing Ads. Many of these platforms have started transitioning towards a first-price auction. For example, in 2021 Google’s AdSense announced plans that they will start using a first-price auction format instead of a second-price one [Won21] due to the simplicity and transparency of first-price (see also [AZ21; Cho+20;

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<sup>\*</sup>Supported in part by AFOSR grant FA9550-19-1-0183 and FA9550-23-1-0068, the Department of Defense (DoD) through the National Defense Science & Engineering Graduate (NDSEG) Fellowship Program, and the Onassis Foundation – Scholarship ID: F ZS 068-1/2022-2023.

<sup>†</sup>Supported in part by NSF grant CCF-1408673, AFOSR grant FA9550-19-1-0183 and FA9550-23-1-0068.

DRS21]). In these platforms, advertisers participate in thousands of auctions per day, and their bidding must take into account the limited budget that they have available. Because of this complexity, advertisers usually deploy automated bidding algorithms that, given the advertiser’s budget and preferences, make bids in these sequential auctions.

A common strategy in many of these automated bidding algorithms is to use a *multiplicative shading factor*. From a single player’s perspective, every round the player shades her true value by some multiplicative factor in  $[0, 1]$ ; this factor depends on her budget and how much of it has been spent in the past rounds. A very successful application of this strategy is the adaptive pacing algorithm of [BG17] in which the authors provide a simple algorithm that iteratively adapts the shading multiplier. They provide many guarantees compared to the best bidding in retrospect for sequential budgeted second-price auctions (see more about their results in Section 2). The simplicity of using shading multipliers makes the strategy of shading values appealing for budgeted players also in first-price auctions, even though bidding with any single shading factor can be far from the best bidding in retrospect in this case.

Strategies that use shading multipliers have been widely used for budget-limited settings. Motivated by their use in practice, Conitzer et al. [Con+18] study the properties of equilibria in second-price auctions, where every player uses a single multiplier to shade her value. [Con+19] study similar properties of shading factors in first-price auctions. More recently, [BKK22] show that one can get a simple equilibrium in budgeted first-price auctions where every player shades her value and the new values are used to form an equilibrium using classic techniques from unbudgeted settings.

The work of [BG17] focuses on guarantees from the player’s perspective in sequential budgeted second-price auctions assuming that the player’s budget grows linearly with time. They offer lower and upper bounds on achievable *competitive ratio*, i.e., what fraction of the optimal in retrospect utility each player can achieve. [Gai+22] focus on guarantees of the [BG17] pacing algorithm for *liquid welfare*<sup>1</sup>. They assume that the players’ values are sampled from the same distribution every round. Under this assumption, they prove that when all players use the pacing algorithm of [BG17] the resulting liquid welfare is within a factor of 2 of the optimal one, and their results hold for both first and second-price auctions.

In this paper, we aim to relax both of the assumptions used by [Gai+22]. First, we do not make any assumptions about the values of the players, except that they are upper bounded by 1. Second, we replace the assumption that they use the pacing algorithm of [BG17] with a more general behavioral assumption. An appealing and weaker behavioral assumption is to assume that players have small regret or (more generally) small competitive ratio in the utility they achieve, without requiring that they use a particular algorithm. [Gai+22] prove that in sequential budgeted second-price auctions, even when players have no-regret, the resulting liquid welfare can be arbitrarily bad compared to the optimal one (see Theorem 5.1 for the result).

In contrast, we focus on sequential budgeted first-price auctions, for which we show that no-regret and even bounded competitive ratio do imply guarantees about liquid welfare. We summarize our main results next.

- Our first and main result in Section 4, is that in sequential budgeted first-price auctions if every player is guaranteed a competitive ratio of  $\gamma$  compared to using the best fixed shading factor (while budget lasts), then the liquid welfare is within a factor of  $\gamma + 1/2 + O(\gamma^{-1})$  of the optimal one (Theorem 4.2). We note that the best fixed shading factor is a weaker benchmark than the best sequence of bids, used in [BG17], allowing a more broad class of algorithms to be used by the players. When  $\gamma = 1$ , the same theorem guarantees a factor of about 2.41, which is close to the bound of 2 of [Gai+22] and [BKK22], but in a less

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<sup>1</sup>Liquid welfare was first introduced in [DP14] and is the standard efficiency metric that generalizes *social welfare* when players are budget constrained: the liquid welfare of a player is the minimum of the value she receives and her budget. The optimal liquid welfare is a reasonable benchmark for the social value of the outcome: one cannot expect players with low budgets to be able to pay to achieve very high values.

constrained setting: players can employ any competitive strategy (not just a specific algorithm), they are not necessarily in equilibrium, and their values can be picked arbitrarily.

The techniques we use to prove these theorems are different from the ones used by [Gai+22], where they assume i.i.d. values across rounds and that players use a specific algorithm. Their proofs focus on which rounds a player bids less than her value, how long this behavior lasts, and how “efficient” her spending in each round is (specifically they prove their algorithm does not overspend while yielding a high utility each round on expectation). We cannot rely on such properties since we assume that values might be adversarially picked and do not know anything about each player’s behavior in each round (just that their resulting utility is competitive). In contrast, we compare the utility achieved by each player to the utility possible for a well-chosen shading factor  $\lambda \in (0, 1)$ . The competitive ratio assumption guarantees that the players’ utility is comparable to what would have been achievable with such a fixed shading. We then need to distinguish two cases, in order to prove that the latter is high. If by using this shading factor  $\lambda$  a player would not have run out of budget, then her utility is high using ideas similar to the classic price of anarchy work (e.g., [ST13]). In contrast, if that multiplier would have led to the player being limited by her budget, then almost all of the budget is used, which again yields high utility: the total value won by the player would have essentially been at least  $1/\lambda$  times her budget.

- In Section 5, we prove two almost matching lower bounds: if every player has competitive ratio  $\gamma$  or worse then the resulting liquid welfare can be  $\max\{\gamma, 2\}$  times the optimal one, for any  $\gamma \geq 1$  (Theorems 5.2 and 5.3).
- In Section 6, we offer a bidding algorithm for sequential first-price auctions with budgets. When players are budgeted and the setting (in our case the players’ values) is picked adversarially, no-regret guarantees are not achievable (this was proven for the second-price setting by [BG17] and their counter-example works in our case as well). In contrast, work focuses on guarantees about the competitive ratio  $\gamma \geq 1$ . We prove that when the total number of rounds is  $T$ , a player with budget  $B$  can achieve a competitive ratio of  $T/B$  with high probability assuming  $B = T^{1/2+\Omega(1)}$ , even when her values and the other players’ bids are adversarially picked (Theorem 6.3). We note that when the budget grows linearly with time, i.e.,  $B = \Omega(T)$ , this is the same competitive ratio that [BG17] achieve in expectation for second-price auctions, which they prove is tight in that case.

To achieve this result we reduce sequential first-price auctions to the Bandits with Knapsack (BwK) setting. For our reduction, we have to overcome a few differences between the two settings. Our action space is continuous but the known results for BwK assume that, as a function of the action, the rewards are concave and the consumption of the resource is convex, which is not true in our case. Additionally, in BwK, an action that would lead to over-consumption of the resource ends the game, while in our setting the bid is adjusted to the remaining budget and the game goes on.

- Finally, in Section 7, we extend our first result to the case when users have submodular valuations instead of additive ones. In this case, we prove that if every player achieves a competitive ratio of  $\gamma$ , then the liquid welfare is within a factor of  $(2 + \gamma + \sqrt{\gamma^2 + 4})/2$  of the optimal one (Theorem 7.1), a bound that is a little worse than our bound of  $\gamma + O(1)$  for the linear case.

## 2 Related Work

Assuming that players’ behavior is based on using a shading multiplier to shade one’s value has recently received a lot of attention. [BG17] have proposed an adaptive pacing mechanism for sequential second-price auctions that shades one’s bid based on the total budget and payment up to that round. From a single user’s perspective, they prove that even if values and prices are picked adversarially, the player can guarantee a competitive ratio of  $\max\{1, \frac{T}{B}\}$  compared to her maximum utility in retrospect, assuming the maximum value she can have each round is 1,  $B$  is her budget, and  $T$  is the number of rounds; they also

prove that this bound is tight. For the case where values and prices are picked from the same distribution every round, they prove a competitive ratio of 1. They also prove that each player achieves a competitive ratio of 1 when every player is using their pacing algorithm and their values are sampled from distributions that satisfy certain properties. In repeated second-price auctions using a fixed shading factor to bid can yield the optimal utility when the player is not budget limited (which is useful in adversarial budgeted settings) or when the player’s values are continuously distributed. While this is not true for first-price, the simplicity of considering shading factors to manage one’s budget motivates using this bidding strategy in this case also, see for example the works of [BKK22; Che+22a; Con+19]. Similar work for bidding in second price auctions has been done in [Che+22b] where they consider mechanisms that use budget throttling instead of pacing multipliers with guarantees similar to [BG17].

[Gai+22] focus on guarantees of the pacing algorithm of [BG17] in sequential budgeted auctions. They prove that when users’ values are picked from the same distribution every round, liquid welfare is within a 2 factor of the optimal one, up to an additive error, sublinear in the number of rounds both in second and first-price auctions. [Gai+22] also prove that the weaker behavioral assumption that players have no-regret or a small competitive ratio is not enough to bound liquid welfare in second-price auctions: even when players have no-regret, the resulting liquid welfare can be arbitrarily bad compared to the optimal one. In contrast, we prove such guarantees are possible in the case of first price.

In the offline setting, [Con+19], as well as the more recent paper of [BKK22] focus on shading based equilibria in budgeted first-price auctions. [BKK22] prove that when every player’s type is sampled from the same distribution if players shade their values and then use those to bid according to standard symmetric first-price auction equilibrium, this produces a symmetric Bayesian Nash equilibrium of a single item auction while observing the budget limit in expectation. They consider this soft budget limit (in expectation only) as when such an equilibrium solution is repeated many times, concentration will help essentially observe the true overall budget. Their work naturally extends value shading, which is one of the several ways budgets are managed in practice, to non-truthful auctions, e.g., see [Con+18], [Con+19], and [Bal+21] (the first and third works focus on second price and the other one on first price).

Similar bounds for value-maximizing players with return-on-spend constraints are proven by Aggarwal, Badanidiyuru, and Mehta [ABM19], as well as by Babaioff et al. [Bab+21] for more general utility measures. Both show that the liquid welfare of a pure Nash equilibrium of the static game is within a factor of 2 of the optimal liquid welfare, when the underlying auction is truthful, e.g., a second-price auction. Deng et al. [Den+21] improve this guarantee by using implicit or explicit information about the players’ values. In contrast, our efficiency results apply to player behavior in a repeated auction setting not only at equilibrium but also without converging to equilibrium. This type of result was very recently found by Lucier et al. [Luc+23] for value-maximizing players under return-on-investment constraints using a specific algorithm: they come up with an adaptive pacing algorithm that when *employed by every player* guarantees no-regret and the resulting liquid welfare is within a factor of 2 of the optimal one.

As mentioned previously, the concept of liquid welfare was introduced in [DP14] and was also used by [ST13] (the latter refers to liquid welfare as effective welfare). [ST13] focus on the price of anarchy of simple mechanisms (first or second-price auctions) but in terms of budgeted players, they compare the resulting social welfare with the optimal liquid welfare, offering a rather unfair comparison, as the former can be arbitrarily bigger than the latter. [DP14] focus on designing mechanisms that maximize the liquid welfare. [FLP19] convert known incentive compatible mechanisms that maximize social welfare for submodular players to incentive compatible mechanisms that maximize liquid welfare with similar guarantees.

Analyzing the outcome of regret-minimizing players in auctions has received attention from the research community recently. [KN22] study repeated auctions with two players who report their (potentially different) value to a regret-minimizing algorithm. They notice that in second-price auctions the dynamics induced might not converge to the bidding of the equilibrium and therefore players might have incentive

to misreport their values to the algorithms. They also prove that this is not the case for first-price auctions. Another similar line of work is that of [Fen+21] and [Den+22], both of which study the convergence to equilibria of players who bid according to mean-based learning algorithms, a category of no-regret learning algorithms. [Fen+21] focuses on second-price, first-price, and multi-position VCG auctions when players have i.i.d. distributions and prove that the bids converge to the canonical Bayes-Nash equilibrium. [Den+22] focus on first-price auctions when players have fixed but different values and study conditions on the players' values that guarantee or not convergence to a Nash equilibrium.

Another interesting series of related works is that of adversarial Bandits with Knapsacks: a budget limited player is trying to maximize her total reward by picking an action each round, where each action has a reward and costs that are subtracted from the budget of each of her resources; rewards and costs are picked by an adversary. The framework was first introduced by [Imm+19], who proved that the player can achieve  $O(d \log T)$  competitive ratio and sublinear regret, both in expectation and high probability, where  $d$  is the number of resources. [KS20] offer an improved  $O(\log d \log T)$  competitive ratio. Another interesting line of work (that precedes the previous one) focuses on the same scenario, but where the rewards and costs are sampled from an unknown (potentially correlated) distribution every round. In contrast to the adversarial setting, here the competitive ratio of 1 is achievable [BKS13]. Very recently, [CCK22] improved the above result when the player's budget is linear in time, that is  $T/B = O(1)$ . Their algorithm achieves 1 competitive ratio in the stochastic setting and  $T/B = O(1)$  competitive ratio in the adversarial one, which is the same bounds as the one [BG17] achieve in second-price auctions. Additionally, in [CCK22] they provide an algorithm for sequential budgeted first-price auctions with stochastic values and prices but their regret term depends on the number of different bids and values that the player is allowed to make, and becomes infinite if either is infinite. An excellent discussion of both works and multi-armed bandits in general can be found in [Sli19].

### 3 Preliminaries and Model

We assume that there are  $n$  players,  $T$  rounds, and one item per round. Every player  $i \in [n]$  has an additive valuation (we will generalize this in Section 7): every round  $t \in [T]$  she has a value  $v_{it}$  for the item being auctioned that round and if she gets allocated the items of rounds  $\mathcal{T} \subseteq [T]$  her total allocated value is

$$V_i = \sum_{t \in \mathcal{T}} v_{it}.$$

We assume that the players' values are bounded:  $v_{it} \in [0, 1]$  for every  $i \in [n]$ ,  $t \in [T]$ .

Unless stated otherwise, we will focus on first-price auctions. This means that in every round  $t$ , each player  $i$  submits a bid  $b_{it}$  and the player with the highest bid wins the item and pays her bid (ties are broken arbitrarily). We denote with  $p_t$  the price of the item, i.e., the highest bid, and with  $d_{it}$  the highest competing bid faced by player  $i$ , i.e.,  $d_{it} = \max_{j \neq i} b_{jt}$ . If player  $i$  gets allocated the items of rounds  $\mathcal{T} \subseteq [T]$ , then her total payment is

$$P_i = \sum_{t \in \mathcal{T}} p_t.$$

We assume that every player  $i$  has a budget  $B_i$  and *budgeted quasi-linear utility*, i.e., her utility when her total value and payment are  $V_i$  and  $P_i$  is

$$U_i = \begin{cases} V_i - P_i, & \text{if } P_i \leq B_i \\ -\infty, & \text{otherwise} \end{cases}.$$



We evaluate the auction system by measuring how well it does at maximizing the *Liquid Welfare*: the liquid welfare of a player  $i$  is

$$LW_i = \min \{B_i, V_i\}$$

and the total liquid welfare is  $LW = \sum_{i \in [n]} LW_i$ . We denote the optimal liquid welfare  $LW^* = \sum_i LW_i^*$ . Additionally, for a subset of players  $X \subseteq [n]$  we denote  $LW_X = \sum_{i \in X} LW_i$  and similarly for  $LW_X^*$ ,  $P_X$ , etc.

### 3.1 The behavioral assumption

We are assuming that players all learn to bid while participating in the auction. When budgeted players are participating in repeated auctions where the values and prices may be adversarially picked, no-regret learning is not possible, but learning algorithms can achieve a bounded competitive ratio. Specifically in this work, we are going to compare a player's resulting utility with the following benchmark: her utility had she bid  $\lambda$  times her value every round, up to her budget, for any  $\lambda \in [0, 1]$ . To formally define this benchmark, we denote with  $\hat{U}_i(\lambda)$  the resulting utility of player  $i$  if she had bid using shading multiplier  $\lambda \in [0, 1]$  every round, constrained by her budget, i.e.,  $\hat{U}_i(\lambda)$  is her utility if her bid on every round  $t$  was

$$\hat{b}_{it} = \min \left\{ \lambda v_{it}, B_i - \sum_{\tau=1}^{t-1} \hat{b}_{i\tau} \mathbb{1} \left[ \text{bid } \hat{b}_{i\tau} \text{ wins against } d_{i\tau} \right] \right\}$$

i.e., every round  $t$  player  $i$ 's bid is the minimum of  $\lambda v_{it}$  and her remaining budget. Our results are going to hold for any tie-breaking rule, which is why we do not explicitly state one above.

Given the above definitions, the benchmark with which we compare a player  $i$ 's utility is the best-fixed shading multiplier, i.e.,  $\sup_{\lambda \in [0,1]} \hat{U}_i(\lambda)$ . We say that player  $i$  has competitive ratio at most  $\gamma \geq 1$  and regret at most  $\text{Reg}$ , if for her resulting utility  $U_i$  it holds that

$$U_i \geq \frac{\sup_{\lambda \in [0,1]} \hat{U}_i(\lambda) - \text{Reg}}{\gamma}.$$

In the special case that  $\gamma = 1$  and  $\text{Reg} = o(T)$ , we say that player  $i$  has no-regret.

We note that the benchmark we use to compare player  $i$ 's resulting utility is much weaker than the one used in previous work, namely by [BG17]. More specifically, the benchmark that they use is the *best sequence of bids*, which is preferable in terms of individual guarantees. In contrast, our liquid welfare guarantee is achieved by requiring the less restrictive behavior described above.

## 4 Guarantees for Total Liquid Welfare

In this section, we prove the guarantee for liquid welfare when all the players have a bounded competitive ratio. We start with a lower bound for the benchmark we use to compare a player's utility, i.e., the utility she gets when she plays according to the optimal in retrospect shading multiplier. The lower bound relates our benchmark to the optimal liquid welfare, which is the first step towards proving that bounded competitive ratio with respect to that benchmark provides a competitive utility for the player.

**Lemma 4.1.** *Fix a player  $i$ , her values  $v_{i1}, \dots, v_{iT}$ , and the price of the items  $p_1, \dots, p_T$ . Let  $O_i \subseteq [T]$  be the items that player  $i$  gets in the allocation that maximizes the total liquid welfare. Let  $c(\lambda) = \frac{-1}{\ln(1-\lambda)}$ . Then for any  $\lambda \in (0, 1)$ , it holds that*

$$\sup_{\mu \in [0,1]} \hat{U}_i(\mu) \geq \min \left\{ \frac{1-\lambda}{\lambda} LW_i^*, c(\lambda) \lambda LW_i^* - c(\lambda) \sum_{t \in O_i} p_t \right\} - 1.$$

To prove the lemma we examine  $\hat{U}_i(\lambda)$  (what happens when player  $i$  uses multiplier  $\lambda$ ) and distinguish two cases. First, if by using  $\lambda$  player  $i$  is at any round budget constrained, then she has spent almost all her budget, up to  $\lambda$ . This allows us to lower bound her utility since every time she wins an item the value she earns from it is at least  $1/\lambda$  times the payment (whether she is budget constrained or not). Second, if the player never becomes budget constrained with multiplier  $\lambda$ , she is not constrained with any  $\mu \leq \lambda$ . This allows us to pick a random multiplier from a distribution and study the utility of the player while ignoring the budget constraint.

*Proof.* Fix  $\lambda$  and  $O_i$  as described above. First, we examine the case where, if the player had used multiplier  $\lambda$  to bid, then she runs out of budget, i.e., at some round  $t$  she is budget constrained and therefore bids less than  $\lambda v_{it}$ . In this case, her total payment is at least  $B_i - \lambda$  and every time she gets an item the value she gets from it is at least  $1/\lambda$  times the price. This proves that in this case

$$\hat{U}_i(\lambda) \geq \left(\frac{1}{\lambda} - 1\right) (B_i - \lambda) = \left(\frac{1}{\lambda} - 1\right) B_i - 1 + \lambda$$

which proves the lemma since  $B_i \geq \text{LW}_i^*$ .

Now we examine the case where player  $i$  would not be budget constrained when using multiplier  $\lambda$ , which also implies she is not budget constrained for any multiplier  $\mu \leq \lambda$ . If player  $i$  uses multiplier  $\mu \in [0, \lambda]$ , then her utility is

$$\hat{U}_i(\mu) \geq (1 - \mu) \sum_{t \in [T]} v_{it} \mathbb{1}[\mu v_{it} > p_t] \geq (1 - \mu) \sum_{t \in O_i} v_{it} \mathbb{1}[\mu v_{it} > p_t] \quad (1)$$

where the first inequality is true because the highest bid that player  $i$  faces in round  $t$  is at most  $p_t$ .

If the multiplier  $\mu$  is picked from the distribution that has probability density function

$$f_\lambda(\mu) = \begin{cases} \frac{c(\lambda)}{1-\mu}, & \text{if } \mu \in [0, \lambda] \\ 0, & \text{otherwise} \end{cases}$$

then taking the expectation of (1) we get

$$\begin{aligned} \mathbb{E}_{\mu \sim f_\lambda(\mu)} [\hat{U}_i(\mu)] &\geq \int_{\mu=0}^{\lambda} (1 - \mu) \sum_{t \in O_i} v_{it} \mathbb{1}[\mu v_{it} > p_t] \frac{c(\lambda)}{1-\mu} d\mu \\ &\geq c(\lambda) \sum_{t \in O_i} v_{it} (\lambda - p_t/v_{it}) \\ &= c(\lambda)\lambda \sum_{t \in O_i} v_{it} - c(\lambda) \sum_{t \in O_i} p_t. \end{aligned}$$

The above proves what we want because  $\sup_{\mu \in [0, \lambda]} \hat{U}_i(\mu) \geq \mathbb{E}_\mu [\hat{U}_i(\mu)]$  and  $\sum_{t \in O_i} v_{it} \geq \text{LW}_i^*$ . ■

We now prove the guarantee for the total liquid welfare, given a bound for every player's competitive ratio. We require the bound for the competitive ratio to be one with high probability (see Theorem 6.3 for the guarantee on the competitive ratio we prove).

**Theorem 4.2.** *Assume that every player  $i$  has competitive ratio at most  $\gamma \geq 1$  and regret  $\text{Reg}$  with probability  $1 - \delta$ . Then*

$$\text{LW} \geq \frac{\text{LW}^* - O(n)(\text{Reg} + 1)}{\gamma + 1/2 + O(1/\gamma)}$$

with probability at least  $1 - n\delta$ , for any  $\delta > 0$ .

**Remark 4.3.** As shown in the proof below, for  $\gamma \leq 1.73$  the Price of Anarchy of the above bound is  $1 + \sqrt{\gamma + 1}$ , which equals about 2.41 for  $\gamma = 1$ .

The theorem requires a competitive ratio with high probability and not in expectation. This is because of the definition of liquid welfare: having high value with high probability implies high liquid welfare, but high expected value does not imply high liquid welfare. The theorem's proof requires partitioning the players into disjoint groups, depending on their value (if it is greater or not than their budget) and their utility (which of the two bounds of Lemma 4.1 holds). After that, we need to pick the correct value of  $\lambda$  that proves the correct bound.

*Proof.* Fix a player  $i$  and recall the notation that  $V_i$ ,  $U_i$ , and  $P_i$  are  $i$ 's resulting value, utility, and payment, respectively. If player  $i$  has competitive ratio at most  $\gamma$  and regret at most  $\text{Reg}$ , it holds that

$$\begin{aligned} V_i &= U_i + P_i \\ &\geq \frac{1}{\gamma} \sup_{\mu \in [0,1]} \hat{U}_i(\mu) - \frac{\text{Reg}}{\gamma} + P_i && \text{(Competitive Ratio)} \\ &\geq \frac{1}{\gamma} \min \left\{ \frac{1-\lambda}{\lambda} \text{LW}_i^*, c(\lambda)\lambda \text{LW}_i^* - c(\lambda) \sum_{t \in O_i} p_t \right\} - \frac{\text{Reg} + 1}{\gamma} + P_i && \text{(Lemma 4.1).} \end{aligned} \quad (2)$$

Now we partition the players into 3 sets:

- $X = \{i : V_i > B_i\}$ .
- $Y = \{i : V_i \leq B_i, \frac{1-\lambda}{\lambda} \text{LW}_i^* \leq c(\lambda)\lambda \text{LW}_i^* - c(\lambda) \sum_{t \in O_i} p_t\}$ .
- $Z = \{i : V_i \leq B_i, \frac{1-\lambda}{\lambda} \text{LW}_i^* > c(\lambda)\lambda \text{LW}_i^* - c(\lambda) \sum_{t \in O_i} p_t\}$ .

Given this partition and that with probability at least  $1 - n\delta$  inequality (2) holds for all players in  $Y$  and  $Z$ , we get three inequalities<sup>2</sup>:

$$\text{LW}_X = B_X \geq \text{LW}_X^*, \quad (3)$$

$$\text{LW}_Y \geq \frac{1-\lambda}{\gamma\lambda} \text{LW}_Y^* + P_Y - |Y| \frac{\text{Reg} + 1}{\gamma}, \quad (4)$$

$$\text{LW}_Z \geq \frac{c(\lambda)\lambda}{\gamma} \text{LW}_Z^* + P_Z - \frac{c(\lambda)}{\gamma} P_{[n]} - |Z| \frac{\text{Reg} + 1}{\gamma} \quad (5)$$

where in the last inequality we used that  $\sum_{i \in Z} \sum_{t \in O_i} p_t \leq \sum_{t \in [T]} p_t = P_{[n]}$ , which follows from the fact that  $O_1, \dots, O_n$  are disjoint. We now solve each one of (3), (4), and (5) for  $\text{LW}_{(\cdot)}^*$ , split  $P_{[n]} = P_X + P_Y + P_Z$ , and add them to get

$$\begin{aligned} &\text{LW}^* - \max \left\{ \frac{\lambda}{1-\lambda}, \frac{1}{c(\lambda)\lambda} \right\} n (\text{Reg} + 1) \\ &\leq \text{LW}_X + \frac{1}{\lambda} P_X + \frac{\gamma\lambda}{1-\lambda} \text{LW}_Y + \left( \frac{1}{\lambda} - \frac{\gamma\lambda}{1-\lambda} \right) P_Y + \frac{\gamma}{c(\lambda)\lambda} \text{LW}_Z + \left( \frac{1}{\lambda} - \frac{\gamma}{c(\lambda)\lambda} \right) P_Z. \end{aligned} \quad (6)$$

We now use the fact that for all  $i$ ,  $0 \leq P_i \leq \text{LW}_i + \text{Reg}/\gamma$ , which follows from  $P_i \leq B_i$  (otherwise player  $i$  would have unbounded negative utility) and  $V_i - P_i \geq -\text{Reg}/\gamma$  (because of the competitive ratio guarantee

<sup>2</sup>Recall the notation  $\text{LW}_X = \sum_{i \in X} \text{LW}_i$ ,  $P_X = \sum_{i \in X} P_i$ , etc.



and that for any  $\mu, \hat{U}(\mu) \geq 0$ ). This proves that

$$\begin{aligned}
& \frac{1}{\lambda} P_X + \left( \frac{1}{\lambda} - \frac{\gamma\lambda}{1-\lambda} \right) P_Y + \left( \frac{1}{\lambda} - \frac{\gamma}{c(\lambda)\lambda} \right) P_Z \\
& \leq \frac{1}{\lambda} \left( \text{LW}_X + |X| \frac{\text{Reg}}{\gamma} \right) + \left( \frac{1}{\lambda} - \frac{\gamma\lambda}{1-\lambda} \right)^+ \left( \text{LW}_Y + |Y| \frac{\text{Reg}}{\gamma} \right) + \left( \frac{1}{\lambda} - \frac{\gamma}{c(\lambda)\lambda} \right)^+ \left( \text{LW}_Z + |Z| \frac{\text{Reg}}{\gamma} \right) \\
& \leq \frac{1}{\lambda} \text{LW}_X + \left( \frac{1}{\lambda} - \frac{\gamma\lambda}{1-\lambda} \right)^+ \text{LW}_Y + \left( \frac{1}{\lambda} - \frac{\gamma}{c(\lambda)\lambda} \right)^+ \text{LW}_Z + \frac{n}{\gamma\lambda} \text{Reg}
\end{aligned} \tag{7}$$

where in the last inequality we aggregated all the regret terms and used the facts that  $\left( \frac{1}{\lambda} - \frac{\gamma\lambda}{1-\lambda} \right)^+ \leq \frac{1}{\lambda}$  and  $\left( \frac{1}{\lambda} - \frac{\gamma}{c(\lambda)\lambda} \right)^+ \leq \frac{1}{\lambda}$ . Plugging (7) into (6) we get

$$\begin{aligned}
& \text{LW}^* - \max \left\{ \frac{\lambda}{1-\lambda}, \frac{1}{c(\lambda)\lambda} \right\} n (\text{Reg} + 1) - \frac{n}{\gamma\lambda} \text{Reg} \\
& \leq \left( 1 + \frac{1}{\lambda} \right) \text{LW}_X + \left( \frac{\gamma\lambda}{1-\lambda} + \left( \frac{1}{\lambda} - \frac{\gamma\lambda}{1-\lambda} \right)^+ \right) \text{LW}_Y + \left( \frac{\gamma}{c(\lambda)\lambda} + \left( \frac{1}{\lambda} - \frac{\gamma}{c(\lambda)\lambda} \right)^+ \right) \text{LW}_Z \\
& = \left( 1 + \frac{1}{\lambda} \right) \text{LW}_X + \max \left\{ \frac{\gamma\lambda}{1-\lambda}, \frac{1}{\lambda} \right\} \text{LW}_Y + \max \left\{ \frac{\gamma}{c(\lambda)\lambda}, \frac{1}{\lambda} \right\} \text{LW}_Z \\
& \leq \max \left\{ 1 + \frac{1}{\lambda}, \frac{\gamma\lambda}{1-\lambda}, \frac{\gamma}{c(\lambda)\lambda} \right\} \text{LW}.
\end{aligned}$$

To get the bound we want, we need to set an appropriate  $\lambda$  in the above inequality. First, if  $\gamma \leq 1.73$ , we set  $\lambda = \lambda_1 = \frac{1}{\sqrt{1+\gamma}}$  (which we get by solving  $1 + \frac{1}{\lambda} = \frac{\gamma\lambda}{1-\lambda}$ ) and get

$$1 + \frac{1}{\lambda_1} = \frac{\gamma\lambda_1}{1-\lambda_1} = 1 + \sqrt{1+\gamma} \geq \frac{\gamma}{c(\lambda_1)\lambda_1}$$

where one can prove that the inequality holds for  $\gamma \leq 1.73$  (see Appendix A).

For  $\gamma \geq 1.73$ , we solve the equation  $1 + \frac{1}{\lambda} = \frac{\gamma}{c(\lambda)\lambda}$ , set  $\lambda_2 = \gamma W \left( -\frac{e^{-2/\gamma}}{\gamma} \right) + 1$ , where  $W(\cdot)$  is the *Lambert W function*<sup>3</sup>, and get

$$1 + \frac{1}{\lambda_2} = \frac{\gamma}{c(\lambda_2)\lambda_2} = 1 + \frac{1}{\gamma W \left( -\frac{e^{-2/\gamma}}{\gamma} \right) + 1} \geq \frac{\gamma\lambda_2}{1-\lambda_2}$$

where one can prove that the inequality holds for  $\gamma \geq 1.73$  (see Appendix A along with a proof that  $0 < \lambda_2 < 1$ ). Combining the two results we prove the theorem after showing that  $1 + \frac{1}{\gamma W \left( -\frac{e^{-2/\gamma}}{\gamma} \right) + 1} = \gamma + 1/2 + O(1/\gamma)$  and the factor in front of the regret term is  $O(n)$  (see Appendix A for the proof of both). ■

## 5 Impossibility Results for the Total Liquid Welfare

In this section, we are going to prove upper bounds for the guaranteed liquid welfare when the players have a competitive ratio of  $\gamma$ . We first include the bound for sequential budgeted second-price auctions of [Gai+22].

<sup>3</sup>The *Lambert W function*  $W(x)$  is the solution to the equation  $ye^y = x$ , [https://en.wikipedia.org/wiki/Lambert\\_W\\_function](https://en.wikipedia.org/wiki/Lambert_W_function).

**Theorem 5.1** ([Gai+22, Proposition D.1]). *In sequential budgeted second-price auctions, for any number of rounds  $T$  there exists an instance with two players both of who have competitive ratio 1 and 0 regret where the resulting liquid welfare is arbitrarily smaller than the optimal one.*

*Proof.* The first player has budget  $B_1 = T\epsilon$  for a small  $\epsilon > 0$  and value  $v_{1t} = 1$  every round. The second one has budget  $B_2 = T$  and value  $v_{2t} = 1$  every round.

The optimal allocation is to give the item to the second player every round, achieving  $LW^* = T$ .

If player 1 bids  $b_{1t} = 1$  every round and player 2 bids  $b_{2t} = 0$ , then the resulting liquid welfare is  $LW = T\epsilon$  and every player has 0 regret: player 1 gets every item for free, while player 2 has no incentive to get any item at price 1. This completes the proof by taking  $\epsilon \rightarrow 0$ , since  $LW/LW^* = \epsilon$ . ■

We now provide an upper bound for  $\gamma = 1$  in first-price auctions. More specifically, we provide an example where the liquid welfare is half the optimal one, even when players have competitive ratio 1 and no-regret.

**Theorem 5.2.** *In sequential budgeted first-price auctions, for every number of rounds  $T$ , there exists an instance where the players have competitive ratio 1, no-regret, and the resulting liquid welfare is arbitrarily close to  $\frac{1}{2}$  times the optimal one.*

*Proof.* There are two players. Fix constant  $\epsilon > 0$ . The first player has value  $v_{1t} = 1$  every round and a total budget of  $B_1 = T\epsilon$  and the second player has value  $v_{2t} = \epsilon$  every round and a total budget of  $B_2 = T\epsilon$ .

The optimal allocation is to give player 1 the items of approximately the first  $T\epsilon$  rounds and give player 2 the item for the rest of the rounds. This results in the optimal liquid welfare being  $LW^* = \min\{T\epsilon, T\epsilon 1\} + \min\{T\epsilon, T(1 - \epsilon)\epsilon\} = T(2\epsilon - \epsilon^2)$ .

In contrast, if every round player 1 bids infinitesimally above  $\epsilon$  and player 2 bids  $\epsilon$ , then neither player has regret and the resulting liquid welfare is  $LW = \min\{T\epsilon, T1\} + \min\{T\epsilon, 0\} = T\epsilon$ . This proves the theorem by noticing that  $LW/LW^* = 1/(2 - \epsilon)$  and taking  $\epsilon \rightarrow 0$ . ■

Finally, we provide another upper bound, which for large  $\gamma$  makes Theorem 4.2 (and also Theorem 7.1 that we present later) tight. More specifically, we show that if players have competitive ratio  $\gamma$  then the resulting liquid welfare can be  $\gamma$  times less than the optimal one.

**Theorem 5.3.** *In sequential budgeted first-price auctions, for every number of rounds  $T$  and  $\gamma \geq 1$ , there is an instance where every player has competitive ratio at most  $\gamma$  and constant regret, and the resulting liquid welfare is almost  $\gamma$  times smaller than the optimal one.*

Intuitively, the theorem's proof is based on the fact that, if there is only one player who gets only a  $1/\gamma$  fraction of  $T$  identical items, then her competitive ratio is  $\gamma$  and the liquid welfare is  $\gamma$  times less than the optimal.

*Proof.* There are two players, neither of which is budget constrained. Player 1 has value  $v_{1t} = 1$  every round and player 2 has value  $v_{2t} = \epsilon = 1/T$  every round. The optimal liquid welfare is  $LW^* = T$ , by giving all the items to player 1.

The bids of the players are the following: for the first  $T/\gamma$  rounds, player 1 bids  $\epsilon$ , and player 2 bids 0. For the rest of the rounds, player 1 bids 0 and player 2 bids  $\epsilon^2$ . Player 1 has a total utility of  $U_1 = (1 - \epsilon)T/\gamma$  and player 2 has  $U_2 = (\epsilon - \epsilon^2)T(1 - 1/\gamma)$ . This outcome yields liquid welfare  $LW = T/\gamma + \epsilon T(1 - 1/\gamma)$  which is  $\frac{1}{\gamma} + \epsilon \frac{\gamma-1}{\gamma} \leq \frac{1}{\gamma} + \epsilon$  fraction of  $LW^*$ .

We are left to prove that the above outcome has competitive ratio at most  $\gamma$  and regret less than 1 for every player. The best allocation for player 1 would have been to get the first  $T/\gamma$  items for free and the rest of the items for a price of  $\epsilon^2$  (note that this results in more utility than using any constant shading multiplier).

This allocation yields utility  $T/\gamma + (1 - \epsilon^2)T(1 - 1/\gamma) = T(1 - \epsilon^2(1 - 1/\gamma))$ . One can prove that this yields competitive ratio  $\gamma$  and regret less than 1 for player 1.

For player 2 the best allocation would have been to get the second batch of items for free. The utility in that case is  $\epsilon T(1 - 1/\gamma)$ . This yields a competitive ratio of 1 and regret  $T\epsilon^2(1 - 1/\gamma) = \epsilon(1 - 1/\gamma) \leq 1$ . ■

## 6 Algorithm for bounded competitive ratio

In this section, we are going to prove that for a player with budget  $B$ , there exists an algorithm that achieves competitive ratio  $T/B$  and sublinear regret with high probability against the best in retrospect shading factor, as needed by Theorem 4.2. Our bound is going to hold for any values of that player and behavior of the other players, even if they are adversarially picked. We note that classic work in online bandit learning with constraints usually focuses on the best in retrospect distribution of actions the player could have taken. However, our results in Section 4 do not require this stronger benchmark and in our setting the two are equivalent when the player's value or the other players' bids are continuously distributed.

Our algorithm will be based on the classic adversarial Bandits with Knapsacks (BwK) setting, first studied by [Imm+19]. Unlike our setting, in BwK the action space is a discrete set. As a first step towards our reduction, we are going to prove that the difference in using two shading multipliers that are very close results in a very small additive error. This will effectively prove that uniformly discretizing the action space  $[0, 1]$  entails a small additive error. Then we are going to reduce the problem of using a finite set of shading multipliers in sequential budgeted first-price auctions to the classic framework of adversarial BwK with very small additive error.

We first prove that using shading multiplier  $\lambda + \epsilon$  instead of  $\lambda$  yields a small additive error, proportional to  $\epsilon$ . We are going to focus on a single arbitrary player, so we are going to drop the  $i$  subscript throughout this section.

**Lemma 6.1.** *Fix any highest bids the other players have submitted, the player's budget  $B \leq T$ , and her values  $v_1, \dots, v_T$ , and let  $\hat{U}(\lambda)$  be the utility of the player had she used multiplier  $\lambda \in [0, 1]$ . Then for any  $\lambda \in [0, 1]$  and  $\epsilon \in [0, 1 - \lambda]$  it holds that*

$$\hat{U}(\lambda + \epsilon) \geq \hat{U}(\lambda) - \frac{T^2\epsilon}{B} - 2.$$

The lemma is proven by examining two cases. First, if using multiplier  $(\lambda + \epsilon)$  does not make the player run out of budget, then using that multiplier instead of  $\lambda$  can only result in a slightly larger payment, at most  $T\epsilon$ . If by using multiplier  $(\lambda + \epsilon)$  the player runs out of budget, then her payment is almost her budget, which yields a high utility since the value of every item she gets is at least  $\frac{1}{\lambda + \epsilon}$  times the price she paid for it. The second case is a bit more complicated because the new multiplier might result in running out of budget much faster than using multiplier  $\lambda$ , which could result in missing out on some items.

*Proof.* We first study the outcome when the player is using multiplier  $\lambda$ . Let  $\mathcal{T} \subseteq [T]$  be the rounds in which the player wins the auction when bidding with multiplier  $\lambda$  and is not budget constrained, i.e., she bids  $\lambda$  times her value and wins the item. If at some round the player wins an item while being budget constrained, then she bids and pays her entire remaining budget. This means that other than the items in  $\mathcal{T}$ , the player wins at most one more item, whose value is at most 1, proving that

$$\hat{U}(\lambda) \leq \sum_{t \in \mathcal{T}} (1 - \lambda)v_t + 1. \tag{8}$$

When the player bids with multiplier  $\lambda + \epsilon$  then she is guaranteed to win every item of rounds  $\mathcal{T}$ , unless she runs out of budget, meaning she either gets utility at least  $(1 - \lambda - \epsilon) \sum_{t \in \mathcal{T}} v_t$  or pays at least  $B - (\lambda + \epsilon)$ .

This means that

$$\hat{U}(\lambda + \epsilon) \geq \begin{cases} (1 - \lambda - \epsilon) \sum_{t \in \mathcal{T}} v_t, & \text{if player gets } \mathcal{T} \text{ without being budget constrained} \\ \left(\frac{1}{\lambda + \epsilon} - 1\right) (B - (\lambda + \epsilon)), & \text{if } (\lambda + \epsilon) \sum_{t \in \mathcal{T}} v_t \leq B, \text{ but does not get } \mathcal{T} \\ \left(\frac{1}{\lambda + \epsilon} - 1\right) (B - (\lambda + \epsilon)), & \text{if } (\lambda + \epsilon) \sum_{t \in \mathcal{T}} v_t > B \end{cases} \quad (9)$$

where the second and third cases come from the fact that every time the player wins an item, her value for it is at least  $\frac{1}{\lambda + \epsilon}$  times the price she pays for it.

For the second case of (9), because  $(\lambda + \epsilon) \sum_{t \in \mathcal{T}} v_t \leq B$ , we have

$$\hat{U}(\lambda + \epsilon) \geq \left(\frac{1}{\lambda + \epsilon} - 1\right) (B - (\lambda + \epsilon)) \geq (1 - \lambda - \epsilon) \sum_{t \in \mathcal{T}} v_t - 1 + \lambda + \epsilon \geq \hat{U}(\lambda) - \epsilon T - 1$$

where the last inequality holds because of (8),  $\sum_t v_t \leq T$ , and  $\lambda + \epsilon \geq 0$ . Because  $B \leq T$ , the inequality above satisfies the lemma and is also similarly proved for the first case of (9).

For the third case of (9), we have that

$$\begin{aligned} \hat{U}(\lambda + \epsilon) &\geq \left(\frac{1}{\lambda + \epsilon} - 1\right) B - 1 + \lambda + \epsilon \\ &\geq \frac{1 - \lambda - \epsilon}{\lambda + \epsilon} \lambda \sum_{t \in \mathcal{T}} v_t - 1 && \left(\lambda \sum_{t \in \mathcal{T}} v_t \leq B \text{ and } \lambda + \epsilon \geq 0\right) \\ &\geq \hat{U}(\lambda) + \left(\frac{1 - \lambda - \epsilon}{\lambda + \epsilon} \lambda - (1 - \lambda)\right) \sum_{t \in \mathcal{T}} v_t - 2 && \text{(using (8))} \\ &= \hat{U}(\lambda) - \frac{\epsilon}{\lambda + \epsilon} \sum_{t \in \mathcal{T}} v_t - 2 \\ &\geq \hat{U}(\lambda) - \frac{T^2 \epsilon}{B} - 2 && \left((\lambda + \epsilon) \sum_{t \in \mathcal{T}} v_t > B, \sum_{t \in \mathcal{T}} v_t \leq T\right) \end{aligned}$$

The above inequality satisfies the lemma and completes the proof.  $\blacksquare$

Now we are going to prove that given any discretization of the action space  $[0, 1]$ , we can use any algorithm from BwK to achieve a low competitive ratio and no-regret in first-price auctions. We first briefly present the BwK framework.

**Definition 6.1** (BwK framework). There are  $K + 1$  actions,  $T$  rounds, and a resource with a total budget of  $B$ . The adversary picks rewards  $(r_{t,k})_{t \in [T], k \in [K]}$  and costs  $(c_{t,k})_{t \in [T], k \in [K]}$  for every round-action pair. The 0-th arm is assumed to have  $c_{t,0} = r_{t,0} = 0$ . On round  $t$ , without observing the rewards and costs of that round, the player picks a (potentially randomized) action  $k_t = 0, \dots, K$ . The game ends after round  $T$  or on round  $T'$  when the player depletes the resource, i.e., when  $\sum_{t=1}^{T'} c_{t,k_t} > B$ . We denote the total reward of the player with  $\text{REW} = \sum_{t=1}^{\min\{T, T'-1\}} r_{t,k_t}$  and with  $\text{OPT}_{\text{FA}}$  the reward of best fixed action in retrospect. We say that the player has competitive ratio  $\gamma$  and regret  $\text{Reg}$  if

$$\text{REW} \geq \frac{\text{OPT}_{\text{FA}} - \text{Reg}}{\gamma}.$$

If the adversary samples the rewards and costs of every round from a distribution independent of other rounds, the setting is *stochastic*. If the adversary picks them arbitrarily before the first round (i.e., without seeing any of the player's actions), then the adversary is called *oblivious*. Finally, if they are picked with knowledge of previous rounds the adversary is called *adaptive*.

**Lemma 6.2.** Let  $\lambda_1, \dots, \lambda_K \in [0, 1]$  be any multipliers and  $\lambda_0 = 0$ . Using any algorithm for BwK that with probability  $1 - \delta$  has competitive ratio  $\gamma$  and regret  $\text{Reg}_T(\delta, K)$  against an adaptive adversary, we can get an algorithm for repeated first-price auctions that uses only multipliers  $\lambda_0, \dots, \lambda_K$  and achieves utility  $U$  for which

$$\mathbb{P} \left[ U \geq \frac{\max_{k=0, \dots, K} \hat{U}(\lambda_k) - \text{Reg}_T(\delta, K) - 2}{\gamma} \right] \geq 1 - \delta$$

even if the values and prices of the items in each round are picked by an adaptive adversary.

*Proof.* To reduce the problem of learning in sequential first-price auctions to BwK, we set<sup>4</sup>  $r_{t,k} = (1 - \lambda_k)v_t \mathbb{1}[\lambda_k v_t > d_t]$  and  $c_{t,k} = \lambda_k v_t \mathbb{1}[\lambda_k v_t > d_t]$ . We then proceed to run the algorithm from the BwK setting on these rewards and costs.

First, we note that there is a small mismatch between the BwK setting we created and the repeated auction setting. A sequence of actions has the same rewards, costs, and remaining budget in both settings, up to the round where in the BwK setting the player runs out of budget. In BwK if the player picks an action that incurs a cost higher than the remaining budget the game stops. In contrast, in the repeated auction setting, if the bid of the player is higher than her remaining budget, then her bid is adjusted, which may or may not end the game by depleting her budget. However, this causes a small mismatch:

- For the difference between the reward of the algorithm in BwK and the utility of the same actions in the repeated auction setting (where the actions after the BwK algorithm runs out of budget are picked arbitrarily), it holds that  $U \geq \text{REW}$ .
- For the two benchmarks,  $\text{OPT}_{\text{FA}}$  and  $\max_k \hat{U}(\lambda_k)$ , it holds  $\max_k \hat{U}(\lambda_k) \leq \text{OPT}_{\text{FA}} + 2$ . The rewards when playing the  $k$ -th arm and using multiplier  $\lambda_k$  are the same up to before the round when the game stops in the BwK setting. On that round and onwards, in the auction setting the player has less than  $\lambda_k$  budget remaining. As long as this remaining budget is used to win items without lowering the player's bid (i.e., the player bids  $\lambda_k v_t$ ), the player can gain at most  $1 - \lambda_k$  additional utility, since the utility she gains is  $\frac{1}{\lambda_k} - 1$  times the price she pays and she has at most  $\lambda_k$  remaining budget. Additionally, she might earn one more item on the round her budget is depleted. This means that after having less than  $\lambda_k$  remaining budget, she may earn up to  $1 - \lambda_k + 1 \leq 2$  utility.

Using the fact that the two above facts are true almost surely and that

$$\mathbb{P} \left[ \text{REW} \geq \frac{\text{OPT}_{\text{FA}} - \text{Reg}_T(\delta, K)}{\gamma} \right] \geq 1 - \delta$$

we get the lemma. ■

Finally, we prove the main result of this section. We combine the two previous lemmas and the algorithm of Castiglioni, Celli, and Kroer [CCK22, Algorithm 1]. We use the result of Fikioris and Tardos [FT23] that guarantees that the algorithm achieves  $T/B$  competitive ratio with high probability, even against an adaptive adversary.

**Theorem 6.3.** Fix a player with budget  $B$ . When after bidding the player gets to know the highest competing bid, there is an algorithm for sequential first-price auctions that for any  $\delta > 0$  achieves utility  $U$  for which

$$\mathbb{P} \left[ U \geq \frac{\sup_{\lambda \in [0,1]} \hat{U}(\lambda) - O\left(\frac{T^{3/2}}{B} \sqrt{\log(T/\delta)}\right)}{T/B} \right] \geq 1 - \delta$$

even if the values and prices of the items are picked by an adaptive adversary.

<sup>4</sup>Here we assume that the player needs to bid strictly above the highest competing bid but our results hold for any tie breaking rule, even if it changes across rounds.

Our result is only meaningful when  $B = T^{1/2+\Omega(1)}$ , because otherwise, the regret is not sublinear. This is reminiscent of the requirement of  $B = \Omega(\sqrt{T})$  that BwK algorithms have (there is also a  $T^2/B$  lower bound on competitive ratio when  $B < \sqrt{T}$  see [Imm+19]). Additionally, our algorithm guarantees a constant competitive ratio when  $B = \Omega(T)$ , which is needed to get constant approximation guarantees for liquid welfare.

*Proof.* Fix a  $K \in \mathbb{N}$  such that  $K = T^2$  and for  $k = 0, 1, \dots, K$  let  $\lambda_k = k/K$ . We use the algorithm of [CCK22; FT23] that achieves competitive ratio  $\gamma = T/B$  and regret  $\text{Reg}_T(\delta, K) = O\left(\frac{T}{B}\sqrt{T \log(TK/\delta)}\right)$  with probability  $1 - \delta$  and Lemma 6.2 to get that with probability at least  $1 - \delta$ ,

$$U \geq \frac{\max_{k=0, \dots, K} \hat{U}(\lambda_k) - O\left(\frac{T}{B}\sqrt{T \log(TK/\delta)}\right)}{T/B} = \frac{\max_{k=0, \dots, K} \hat{U}(\lambda_k) - O\left(\frac{T^{3/2}}{B}\sqrt{\log(T/\delta)}\right)}{T/B}$$

Fix any multiplier  $\lambda \in [0, 1]$  and let  $\lambda_k$  be such that  $\lambda \leq \lambda_k \leq \lambda + 1/K$ . Using Lemma 6.1 we get that  $\hat{U}(\lambda_k) \geq \hat{U}(\lambda) - O\left(\frac{T^2}{KB}\right) - 2 = \hat{U}(\lambda) - O(1)$ . This proves that  $\max_k \hat{U}(\lambda_k) \geq \sup_\lambda \hat{U}(\lambda) - O(1)$  and completes the proof of the theorem. ■

**Remark 6.4.** We note that even though Theorem 6.3 proves a high probability  $T/B$  competitive ratio and sub-linear regret for sequential budgeted first-price auctions when  $B = T^{1/2+\Omega(1)}$ , our proof can be easily adapted to use the guarantee of any BwK algorithm, either with high probability or in expectation, in the stochastic or adversarial case. For example, for certain values of  $B$ , we could get the  $O(\log T)$  competitive ratio guarantee that [Imm+19] achieve against an oblivious adversary. In addition, when the player has bandit feedback (i.e., only observes the value and payment of her action) then we can get the same competitive ratio with slightly higher regret.

## 7 Submodular Valuations

We now move to the final section of our results, where we generalize the results of Section 4 for the case where players have submodular valuations. If player  $i$  receives a bundle of items  $\mathcal{T} \subseteq [T]$  then her value is  $v_i(\mathcal{T})$ , where  $v_i$  is a submodular<sup>5</sup>, non-decreasing, and non-negative function. We use the standard notation for the marginal value of item  $t$  for bundle  $\mathcal{T}$ :  $v_i(t | \mathcal{T}) = v_i(\mathcal{T} \cup \{t\}) - v_i(\mathcal{T})$ . The definitions of the players' utilities and liquid welfare remain the same.

Before bounding the liquid welfare in this setting, we first define how bidding according to a multiplier  $\lambda \in [0, 1]$  works in this case. If player  $i$  in round  $t$  uses multiplier  $\lambda$  and has already gained items  $\mathcal{T} \subseteq [t-1]$  then she bids  $\lambda$  her current marginal value for item  $t$ ,  $\lambda v_i(t | \mathcal{T})$ , as long as she is not budget constrained (if she is budget constrained she bids her remaining budget). Because the marginal value of the item in every round (and therefore the bid) of every player depends on her past allocation, this setting is more complicated than the one we studied in previous sections. Most notably, it is not clear if there exists an algorithm with bounded competitive ratio and regret, as we showed in Section 6 for the additive case. The reason for that is that a reduction to BwK is much harder since the reward and consumption of the resource in a single round depend on the results of the previous ones. We leave the last question as future work.

In this section, we prove the following theorem, which proves a slightly worse bound than the one of Theorem 4.2.

<sup>5</sup>A set function  $v : 2^{[T]} \rightarrow \mathbb{R}$  is submodular if for any  $S \subseteq \mathcal{T} \subseteq [T]$  and  $t \notin \mathcal{T}$  it holds that  $v(S \cup \{t\}) - v(S) \geq v(\mathcal{T} \cup \{t\}) - v(\mathcal{T})$



**Theorem 7.1.** Assume that every player  $i$  has competitive ratio at most  $\gamma \geq 1$  and regret  $\text{Reg}$  with probability  $1 - \delta$ . If the players have submodular valuations then

$$\text{LW} \geq \frac{\text{LW}^* - O(n)(\text{Reg} + 1)}{\frac{2 + \gamma + \sqrt{\gamma^2 + 4}}{2}}$$

with probability at least  $1 - n\delta$ , for any  $\delta \in (0, 1/n)$ .

**Remark 7.2.** We note that the factor in the denominator in Theorem 7.1 for  $\gamma = 1$  is about 2.62 and for all  $\gamma$  is a bit bigger than the one in Theorem 4.2. Both asymptotically are  $\gamma + O(1)$ . We give a plot of both for small  $\gamma$  in Fig. 1.

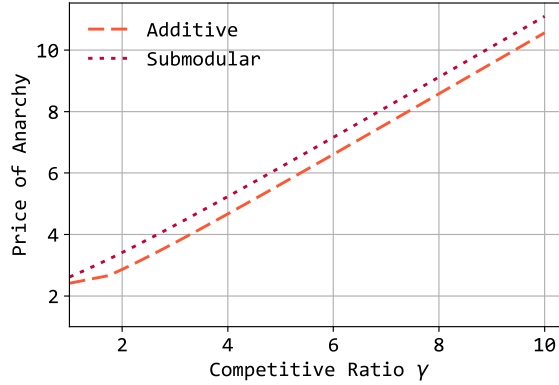


Figure 1: Price of Anarchy plots for Theorems 4.2 and 7.1 for  $\gamma \in [1, 10]$ .

We start with a simple lemma that will help us lower bound the value of the bundle  $\mathcal{T}_i$  gained by player  $i$  when she uses a fixed shading multiplier  $\lambda$ . More specifically, we know that if the player is not budget constrained in round  $t$ , then  $t \notin \mathcal{T}_i$  implies  $\lambda v_i(t | \mathcal{T}_i \cap [t-1]) \leq p_t$ . In the following lemma, we lower bound  $v_i(\mathcal{T}_i)$  for any  $\mathcal{T}_i$  that satisfies the previous implication.

**Lemma 7.3.** Fix a player  $i$ , her submodular valuation  $v_i : 2^{[T]} \rightarrow \mathbb{R}_{\geq 0}$ , and the resulting prices of the items  $p_1, \dots, p_T$ . Let  $\lambda \in (0, 1]$  and  $\mathcal{T}_i \subseteq [T]$  such that

$$t \notin \mathcal{T}_i \implies v_i(t | \mathcal{T}_i \cap [t-1]) \leq \frac{1}{\lambda} p_t.$$

Then, for any set  $O_i \subseteq [T]$  it holds that

$$v_i(\mathcal{T}_i) \geq v_i(O_i) - \frac{1}{\lambda} \sum_{t \in O_i} p_t.$$

*Proof.* We have that

$$\begin{aligned} v_i(\mathcal{T}_i) + \frac{1}{\lambda} \sum_{t \in O_i} p_t &\geq v_i(\mathcal{T}_i) + \frac{1}{\lambda} \sum_{t \in O_i \setminus \mathcal{T}_i} p_t && (p_t \geq 0) \\ &\geq v_i(\mathcal{T}_i) + \sum_{t \in O_i \setminus \mathcal{T}_i} v_i(t | \mathcal{T}_i \cap [t-1]) && (t \notin \mathcal{T}_i) \\ &\geq v_i(\mathcal{T}_i) + \sum_{t \in O_i \setminus \mathcal{T}_i} v_i(t | \mathcal{T}_i \cup (O_i \cap [t-1])) && (\text{submodularity}) \\ &= v_i(O_i \cup \mathcal{T}_i) \geq v_i(O_i) \end{aligned}$$

where in the last equality, every term in the sum iteratively adds the marginal of an item  $t$  that is in  $O_i$  and not in  $\mathcal{T}_i$ .  $\blacksquare$

Next, we prove a lemma analogous to Lemma 4.1, where we lower bound the utility of a player if she used a fixed shading multiplier. The following lemma provides a worse guarantee because the lemma does not use randomization on the multiplier picked.

**Lemma 7.4.** *Fix a player  $i$  and the resulting prices of the items  $p_1, \dots, p_T$ . Then for any  $\lambda \in (0, 1)$  it holds that*

$$\hat{U}_i(\lambda) \geq (1 - \lambda) \left( \text{LW}_i^* - \frac{1}{\lambda} \sum_{t \in O_i} p_t \right) - 1$$

where  $O_i \subseteq [T]$  is the bundle player  $i$  gets in the allocation that maximizes the total liquid welfare.

*Proof.* Assume that by using multiplier  $\lambda$  player  $i$  would have gotten bundle  $\mathcal{T}_i$  if she was never budget constrained; in that case she would have had utility  $(1 - \lambda)v_i(\mathcal{T}_i)$  and it would have held  $\lambda v_i(\mathcal{T}_i) \leq B_i$ . If she was budget constrained (in which case  $\lambda v_i(\mathcal{T}_i) > B_i$ ) she would have spent at least  $B_i - \lambda$ , which yields a utility of at least  $(1/\lambda - 1)(B_i - \lambda)$ , since the marginal value she gets from any item she wins is at least  $1/\lambda$  what she pays for it. This proves

$$\begin{aligned} \hat{U}_i(\lambda) &\geq \begin{cases} (1 - \lambda)v_i(\mathcal{T}_i), & \text{if } \lambda v_i(\mathcal{T}_i) \leq B_i \\ (\frac{1}{\lambda} - 1)(B_i - \lambda), & \text{if } \lambda v_i(\mathcal{T}_i) > B_i \end{cases} \\ &\geq (1 - \lambda) \min \left\{ v_i(\mathcal{T}_i), \frac{1}{\lambda} B_i - 1 \right\}. \end{aligned}$$

Because the bundle  $\mathcal{T}_i$  satisfies the requirements of Lemma 7.3 the above becomes

$$\hat{U}_i(\lambda) \geq (1 - \lambda) \min \left\{ v_i(O_i) - \frac{1}{\lambda} \sum_{t \in O_i} p_t, \frac{1}{\lambda} B_i - 1 \right\}.$$

The above proves what we want since  $v_i(O_i) \geq \text{LW}_i^*$ ,  $p_t \geq 0$ , and  $\frac{1}{\lambda} B_i \geq \frac{1}{\lambda} \text{LW}_i^* \geq \text{LW}_i^*$ .  $\blacksquare$

Now we proceed to prove Theorem 7.1, by picking the right multiplier  $\lambda$ . The proof is similar to the one in Theorem 4.2.

*Proof of Theorem 7.1.* Fix a player  $i$  and recall the notation that  $V_i$ ,  $U_i$ , and  $P_i$  are  $i$ 's resulting value, utility, and payment, respectively. If player  $i$  has competitive ratio at most  $\gamma$  and regret at most  $\text{Reg}$ , it holds that

$$\begin{aligned} V_i &= U_i + P_i \\ &\geq \frac{1}{\gamma} \sup_{\mu \in [0, 1]} \hat{U}_i(\mu) - \frac{\text{Reg}}{\gamma} + P_i && \text{(Regret)} \\ &\geq \frac{1 - \lambda}{\gamma} \left( \text{LW}_i^* - \frac{1}{\lambda} \sum_{t \in O_i} p_t \right) - \frac{\text{Reg} + 1}{\gamma} + P_i && \text{(Lemma 7.4)} \end{aligned}$$

Now we partition the players into 2 sets:

1.  $X = \{i : V_i > B_i\}$ .
2.  $Y = \{i : V_i \leq B_i\}$ .

Given this partition and with probability at least  $1 - n\delta$  the above inequality holds for all players in  $Y$  we get two inequalities<sup>6</sup>:

$$LW_X = B_X \geq LW_X^*, \quad (10)$$

$$LW_Y \geq \frac{1-\lambda}{\gamma} LW_Y^* + P_Y - \frac{1-\lambda}{\gamma\lambda} P_{[n]} - |Y| \frac{\text{Reg} + 1}{\gamma} \quad (11)$$

where in the last inequality we used the fact that  $O_1, \dots, O_n$  are disjoint.

Now solving (11) for  $LW_Y^*$  and adding it with (10) we get

$$\begin{aligned} & LW^* - \frac{1}{1-\lambda} n (\text{Reg} + 1) \\ & \leq LW_X + \frac{1}{\lambda} P_X + \frac{\gamma}{1-\lambda} LW_Y + \left( \frac{1}{\lambda} - \frac{\gamma}{1-\lambda} \right) P_Y \end{aligned}$$

We now use the fact that  $0 \leq P_i \leq LW_i + \text{Reg}/\gamma$ , which follows from  $P_i \leq B_i$  and  $V_i - P_i \geq -\text{Reg}/\gamma$  (both following because player  $i$  has bounded competitive ratio). This makes the above inequality

$$\begin{aligned} & LW^* - \frac{1}{1-\lambda} n (\text{Reg} + 1) - \frac{n}{\gamma\lambda} \text{Reg} \\ & \leq \left( 1 + \frac{1}{\lambda} \right) LW_X + \left( \frac{\gamma}{1-\lambda} + \left( \frac{1}{\lambda} - \frac{\gamma}{1-\lambda} \right)^+ \right) LW_Y \\ & = \left( 1 + \frac{1}{\lambda} \right) LW_X + \max \left\{ \frac{\gamma}{1-\lambda}, \frac{1}{\lambda} \right\} LW_Y \\ & \leq \max \left\{ 1 + \frac{1}{\lambda}, \frac{\gamma}{1-\lambda} \right\} LW. \end{aligned}$$

To get the bound we want, we need to set an appropriate  $\lambda$  in the above inequality. We set  $\lambda = \frac{2}{\gamma + \sqrt{\gamma^2 + 4}}$  which makes the above inequality

$$LW^* - \frac{1 + \gamma + \sqrt{\gamma^2 + 4}}{\gamma} n (\text{Reg} + 1) \leq \frac{2 + \gamma + \sqrt{\gamma^2 + 4}}{2} LW$$

and proves the theorem since  $\frac{1 + \gamma + \sqrt{\gamma^2 + 4}}{\gamma} \leq 2 + \sqrt{5}$ . ■

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<sup>6</sup>Recall the notation  $LW_X = \sum_{i \in X} LW_i$ ,  $P_X = \sum_{i \in X} P_i$ , etc.

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## A Calculations for the proof of Theorem 4.2

We first show that  $\lambda_1$  and  $\lambda_2$  are in the range  $(0, 1)$ . The first is obvious, since  $\lambda_1 = \frac{1}{\sqrt{\gamma+1}}$ . To prove that  $\lambda_2 = \gamma W(-e^{-2/\gamma}/\gamma) + 1 \in (0, 1)$ , we first note that  $W(x)$  is an increasing function in  $x$  and that  $W(x) \in (-1, 0)$  when  $x \in (-1/e, 0)$ . This immediately proves that  $\lambda_2 < 1$ . We now notice that

$$W\left(-e^{-2/\gamma}/\gamma\right) > W\left(-e^{-1/\gamma}/\gamma\right) = -\frac{1}{\gamma}$$

where the first inequality follows from the fact that  $W$  is strictly increasing and the second from the fact that  $W(xe^x) = x$ , by definition for  $x \geq -1$ . This proves that  $\lambda_2 > 0$ .

Now let  $\gamma_0$  be the solution to

$$1 + \frac{1}{\lambda_1} = \frac{\gamma}{c(\lambda_1)\lambda_1} \iff 1 + \sqrt{\gamma+1} = -\gamma\sqrt{\gamma+1} \ln\left(1 - \frac{1}{\sqrt{\gamma+1}}\right)$$

We prove that  $\gamma_0$  is the unique solution by noticing that  $1 + \sqrt{\gamma+1} + \gamma\sqrt{\gamma+1} \ln\left(1 - \frac{1}{\sqrt{\gamma+1}}\right)$  is strictly decreasing for  $\gamma \geq 1$  (can be proved in Mathematica). Numerically we can solve the above equation to see that  $\gamma_0 \approx 1.73$ . These observations also prove that for  $\gamma < \gamma_0$

$$1 + \frac{1}{\lambda_1} > \frac{\gamma}{c(\lambda_1)\lambda_1}$$

as needed by the proof in Theorem 4.2.

We now prove that for  $\gamma \geq \gamma_0$  it holds that

$$1 + \frac{1}{\lambda_2} \geq \frac{\gamma\lambda_2}{1 - \lambda_2} \tag{12}$$



First, we notice that the equality holds when  $\gamma = \gamma_0$ , since from before this was the unique solution to the system  $1 + \frac{1}{\lambda} = \frac{\gamma\lambda}{1-\lambda} = \frac{\gamma}{c(\lambda)\lambda}$ . To prove the inequality all we need to prove is that  $1 + \frac{1}{\lambda_2}$  is increasing in  $\gamma$  and  $\frac{\gamma\lambda_2}{1-\lambda_2}$  is decreasing in  $\gamma$ . We do both of these by proving that  $\lambda_2$  is decreasing in  $\gamma$ . The derivative of  $\lambda_2$  (calculated in Mathematica) is

$$\frac{d\lambda_2}{d\gamma} = \frac{W\left(-\frac{e^{-2/\gamma}}{\gamma}\right) 2 + \gamma W\left(-\frac{e^{-2/\gamma}}{\gamma}\right)}{\gamma \left(1 + W\left(-\frac{e^{-2/\gamma}}{\gamma}\right)\right)}.$$

The above quantity is strictly negative, because of the observations made at the beginning of the section:

- $W\left(-\frac{e^{-2/\gamma}}{\gamma}\right) \in (-1, 0)$ .
- $2 + \gamma W\left(-\frac{e^{-2/\gamma}}{\gamma}\right) = 1 + \lambda_2 \in (1, 2)$ .

These two observations prove (12).

We now point out that  $1 + 1/\lambda_2 = \gamma + 1/2 + O(1/\gamma)$ . One can verify that this is the case using Mathematica. Finally, we need to bound the factor in front of the regret term for  $\lambda \in \{\lambda_1, \lambda_2\}$ ,

$$\max\left\{\frac{\lambda}{1-\lambda}, \frac{1}{c(\lambda)\lambda}\right\} + \frac{1}{\gamma\lambda}$$

We use the fact proven above, that for  $\lambda \in \{\lambda_1, \lambda_2\}$ ,

$$1 + \frac{1}{\lambda} = \max\left\{1 + \frac{1}{\lambda}, \frac{\gamma\lambda}{1-\lambda}, \frac{\gamma}{c(\lambda)\lambda}\right\}$$

to get that

$$\max\left\{\frac{\lambda}{1-\lambda}, \frac{1}{c(\lambda)\lambda}\right\} + \frac{1}{\gamma\lambda} \leq \frac{1}{\gamma} + \frac{1}{\gamma\lambda} + \frac{1}{\gamma\lambda} = O(1)$$

where the final equality holds because either  $\lambda = \lambda_1 = \Theta(1/\sqrt{\gamma})$  or for  $\lambda = \lambda_2 = \Theta(1/\gamma)$