An Axiomatization of the Pairwise Netting Proportional Rule in Financial Networks*

Péter Csóka† P. Jean-Jacques Herings‡

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Abstract

We consider financial networks where agents are linked to each other via mutual liabilities. In case of bankruptcy, there are potentially many bankruptcy rules, ways to distribute the assets of a bankrupt agent over the other agents. One common approach is to first apply pairwise netting of agents that have mutual liabilities and next use the proportional rule to determine the payments on the basis of the net liabilities. We refer to this as the pairwise netting proportional rule. The pairwise netting proportional rule satisfies the basic requirements of claims boundedness, limited liability, priority of creditors, and continuity. It also satisfies the desirable properties of net impartiality, an agent that has two creditors with the same net claims pays the same amount to both creditors on top of pairwise netting, and invariance to mitosis, an agent that splits into a number of identical agents is not affecting the payments of the other agents. We demonstrate that if net impartiality and invariance to mitosis, together with the basic requirements, are regarded as imperative properties, then payments should be determined by the pairwise netting proportional rule.

Keywords: Financial networks, systemic risk, portfolio compression, clearing, pairwise netting.

JEL Classification: C71, G10

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†Institute of Finance, Corvinus University of Budapest and Centre for Economic and Regional Studies. E-mail: peter.csoka@uni-corvinus.hu.

‡Department of Econometrics and Operations Research, Tilburg University, P.O. Box 90153, 5000 LE, Tilburg, The Netherlands. E-mail: P.J.J.Herings@tilburguniversity.edu.

1 Introduction

Financial crises, again and again, draw attention to the importance of financial networks. In a financial network, bankruptcy can also occur due to contagion, where agents default due to not being able to collect all of their claims. We consider financial networks where agents have fully liquid initial endowments and are linked to each other with liabilities. Payments are determined endogenously and are given by bankruptcy rules. The most often used bankruptcy rule in financial networks is the proportional rule, where payments are proportional to liabilities, as in the seminal paper Eisenberg and Noe (2001) as well as in extensions of the basic model as presented in Cifuentes, Ferrucci, and Shin (2005), Shin (2008), Rogers and Veraart (2013), Csóka and Herings (2018), Demange (2018), and Schuldenzucker, Seuken, and Battiston (2020). For excellent surveys of the literature, we refer to Glasserman and Young (2016) and Jackson and Pernoud (2021).

An alternative to the proportional rule is to perform one round of pairwise netting of mutual liabilities first and apply the proportional rule to the net liabilities next. We call the corresponding bankruptcy rule the pairwise netting proportional rule. Pairwise netting, also called bilateral netting or bilateral compression, is most often applied to over-the-counter derivatives trades for contracts like forwards, options, swaps, and credit derivatives, see Duffie and Zhu (2011), Zawadowski (2013), Cont and Kokholm (2014), Amini, Filipović, and Minca (2016), Garratt and Zimmerman (2020), and D'Errico and Roukny (2021).

A drawback of the proportional rule is that agents with strictly positive initial endowments and strictly higher claims than liabilities towards all other agents may still end up with zero equity and default under the proportional rule. We demonstrate that they will always have positive equity and be solvent under the pairwise netting proportional rule. These observations provide a strong argument in favor of the latter bankruptcy rule.

The main contribution of this paper is to provide an axiomatization of the pairwise netting proportional rule. Our main substantive axioms are net impartiality and invariance to mitosis. Net impartiality requires that if agent i has the same net liability to agent j and agent k, then agent i should make the same net payment to agents j and k. Invariance to mitosis requires that the split of an agent into multiple identical agents should not affect the payments made to and received from agents not involved in the split. Additionally, we impose the basic axioms of claims boundedness, limited liability, priority of creditors, and continuity. Claims boundedness requires that all payments should be bounded from above by the respective liabilities. Limited liability requires that all agents should end up with non-negative equity. Priority of creditors requires that all defaulting agents should end up with zero equity. Continuity requires the bankruptcy rule to be continuous. We show these axioms to be independent.

Csóka and Herings (2021) imposes impartiality rather than net impartiality. Impartiality requires that if agent i has the same liability to both agents j and k, then agent i should make the same payment to agents j and k. Csóka and Herings (2021) show that impartiality together with the other axioms characterizes the proportional rule in financial networks.

Contrary to net impartiality, impartiality does not take the liabilities of agents j and k towards agent i into account, and may therefore be used in the characterization of bankruptcy rules for network problems that are directly derived from division rules for the simpler class of claims problems, the class of problems where agents have claims on a single, fixed, estate. Almost all bankruptcy rules that are axiomatized in the setting of financial networks are of this type, like the axiomatization of equity resulting from the Aumann-Maschler rule in Groote Schaarsberg, Reijnierse, and Borm (2018) and the axiomatization of equity resulting from the constrained equal losses, constrained equal awards, and proportional rule by Ketelaars and Borm (2021), which extends the axiomatization for the related claims rules in Moulin (2000). An exception is Demange (2022), who defines and axiomatizes the constrained-proportional bankruptcy rule, a bankruptcy rule that is also not directly derived from division rules. The pairwise netting proportional bankruptcy rule in this paper can also not be directly derived from division rules for claims problems and the possibility of mutual claims in a financial network plays a substantive role in the axiomatization.

The axiomatization of bankruptcy rules in the setting of claims problems has a long tradition in the literature, starting from the seminal paper by O'Neill (1982). For axiomatizations of the proportional rule in different models for claims problems, see Young (1988), de Frutos (1999), Tasnádi (2002), Moreno-Ternero (2006), and Ju, Miyagawa, and Sakai (2007), Moulin (2016), and Thomson (2016). For a comprehensive overview of this literature, see Thomson (2019). The issue of whether impartiality should be imposed on absolute liabilities or on net liabilities is genuine to the network setting. In the class of claims problems, there is no difference between these two axioms, since netting has no effect on the liabilities. As a consequence, all the well-known division rules that have been suggested for the class of claims problems, like the proportional rule, the constrained equal awards rule, or the Talmud rule to name a few, satisfy both impartiality and net impartiality.

The paper is organized as follows. In Section 2, we define financial networks, the proportional rule, and the pairwise netting proportional rule and illustrate them with examples. Section 3 provides definitions and a discussion of the axioms. In Section 4, we characterize the pairwise netting proportional rule as the solution to a system of equations and inequalities. The axiomatization can be found in Section 5. In Section 6 we show that the axioms are independent. Section 7 contains the conclusion.

2 Financial Networks and Pairwise Netting

In this section, we present the standard framework of a financial network, which is based on the seminal work by Eisenberg and Noe (2001).

We consider a countably infinite set of potential agents, without loss of generality represented by the set of natural numbers $\mathbb{N}$, and denote the collection of non-empty, finite subsets of $\mathbb{N}$ by $\mathcal{N}$.

A *financial network* $F = (N, z, L)$ consists of a set of agents $N \in \mathcal{N}$, a strictly positive vector $z \in \mathbb{R}^N_{++}$ of *initial endowments*, and a non-negative *liability matrix* $L \in \mathbb{R}^{N \times N}_{+}$. The set of all financial networks is denoted by $\mathcal{F}$.

The initial endowments consist of all the assets of an agent, except the claims the agent has on the other agents, which are part of the liability matrix L. Entry L_{ij} of the liability matrix represents the claim of agent j on agent i or, equivalently, the liability of agent i towards agent j. As a notational convention, we assume that $L_{ii} = 0$, so agents do not hold claims on themselves. It is allowed that simultaneously $L_{ij} > 0$ and $L_{ji} > 0$, so agent i can have a liability towards agent j and at the same time agent j can have a liability towards agent i. In fact, the question of how such cases should be treated is at the heart of the contribution of this paper.

The set of all matrices in $\mathbb{R}^{N \times N}_{+}$ with a zero diagonal is denoted by $\mathcal{M}(N)$. Row $i \in N$ of a matrix $P \in \mathcal{M}(N)$ is denoted by P_i and column $i \in N$ of this matrix by P^i. The union over all sets in $\mathcal{N}$ of such matrices is denoted by $\mathcal{M} = \cup_{N \in \mathcal{N}} \mathcal{M}(N)$.

In principle, agents are required to settle their liabilities by making mutual payments equal to those liabilities. But this is only feasible if the financial network (N, z, L) is such that, for every $i \in N$, $z_i + \sum_{j \in N} L_{ji} \geq \sum_{j \in N} L_{ij}$, i.e., for every agent i, the initial endowments z_i plus the sum of the claims on the other agents $\sum_{j \in N} L_{ji}$ is sufficient to settle the liabilities $\sum_{j \in N} L_{ij}$ with all the other agents. As soon as there is a single agent $i \in N$ such that $z_i + \sum_{j \in N} L_{ji} < \sum_{j \in N} L_{ij}$, bankruptcy is bound to occur, at least one agent cannot settle some of the liabilities with some of the other agents, and bankruptcy law determines the payments to be made. These payments are collected in a *payment matrix* $P \in \mathcal{M}(N)$, where P_{ij} is the monetary amount to be paid by agent $i \in N$ to agent $j \in N$.

Given a payment matrix $P \in \mathcal{M}(N)$, the *asset value* $a_i(N, z, P)$ of agent $i \in N$ is given by

$$a_i(N, z, P) = z_i + \sum_{j \in N} P_{ji}.$$

The *equity* $e_i(N, z, P)$ of an agent $i \in N$ is given by

$$e_i(N, z, P) = a_i(N, z, P) - \sum_{j \in N} P_{ij} = z_i + \sum_{j \in N}(P_{ji} - P_{ij}),$$

so subtracting the payments as made by an agent from the asset value of the agent yields the agent's equity.

A function that assigns a payment matrix $P \in \mathcal{M}(N)$ to each financial network $(N, z, L) \in \mathcal{F}$ is called a bankruptcy rule, more formally defined as follows.

Definition 2.1. A *bankruptcy rule* is a function $b : \mathcal{F} \to \mathcal{M}$ such that for every $(N, z, L) \in \mathcal{F}$ it holds that $b(N, z, L) \in \mathcal{M}(N)$.

An important principle in bankruptcy law is proportionality. In case of insolvency of an agent, the agent makes payments in proportion to the agent's liabilities. To define this more formally, it is helpful to introduce the non-negative *relative liability matrix* $\Pi \in \mathbb{R}_+^{N \times N}$, where for $i, j \in N$,

$$\Pi_{ij} = \begin{cases} 0, & \text{if } L_{ij} = 0, \\ \frac{L_{ij}}{\sum_{k \in N} L_{ik}}, & \text{otherwise.} \end{cases}$$

Definition 2.2. The *proportional rule* is the function $b^{\mathrm{p}} : \mathcal{F} \to \mathcal{M}$ such that for every $(N, z, L) \in \mathcal{F}$ it holds that $b^{\mathrm{p}}(N, z, L) = P$, where the matrix $P \in \mathcal{M}(N)$ solves the following system of equations:

$$P_{ij} = \min\left\{\Pi_{ij}(z_i + \sum_{k \in N} P_{ki}), L_{ij}\right\}, \quad i, j \in N. \tag{2.1}$$

Under the proportional rule, a proportion Π_{ij} of the asset value $a_i(P) = z_i + \sum_{k \in N} P_{ki}$ of every agent $i \in N$ is spent to settle the liability with agent $j \in N$, up to the maximum of L_{ij}. It follows from Theorem 2 in Eisenberg and Noe (2001) that the system of equations (2.1) has a unique solution, so the function b^{p} is well-defined.

We illustrate financial networks and the proportional rule in Example 2.3.

Example 2.3 (The proportional rule)**.** Consider the financial network $(N, z, L) \in \mathcal{F}$ with three agents $N = \{1, 2, 3\}$ and initial endowments and liabilities as in the first two blocks of Table 1. Table 1 also presents the payment matrix P resulting from the proportional rule b^{p}, the induced asset values $a(N, z, P)$, and equities $e(N, z, P)$.

Agents 1 and 2 default on their liabilities and end up with zero equity. As a result, all initial endowments, with a total value of 51, are transferred to agent 3. Notice that even if agent 1 fully received the claim on agent 2, the resulting asset value of $10 + 360 = 370$ would not be sufficient to pay all liabilities, which are equal to $400 + 40 = 440$. Agent 1 is

z	L			P			$a(N,z,P)$	$e(N,z,P)$
10	0	400	40	0	320	32	352	0
1	360	0	20	342	0	19	361	0
40	0	40	0	0	40	0	91	51

Table 1: The initial endowments, the liabilities, the payment matrix, the asset values, and the equities resulting from the proportional rule b^{p} in Example 2.3.

therefore in fundamental default. Agent 2 also defaults, but if agent 2 received the entire claims on agents 1 and 3, the resulting asset value of $1+400+40 = 441$ would be sufficient to pay the total liabilities of $360 + 20 = 380$. The default of agent 2 is therefore due to contagion.

The insolvency of agent 2 is particularly disturbing. Agent 2 has a claim of 400 on agent 1, which is more than the liability of 360 towards agent 1, and a claim of 40 on agent 3, which exceeds the liability of 20 towards agent 3. Moreover, agent 2 has positive initial endowments. Still, agent 2 goes bankrupt and ends up with zero equity when payments are determined by the proportional rule.

Example 2.3 displays another unappealing feature of the proportional rule. One could argue that the common part of the mutual liabilities of two agents $i, j \in N$, i.e., the minimum of L_{ij} and L_{ji}, should be irrelevant, since agents i and j can decrease or increase this amount at will without affecting their real financial situation. The net liabilities of agent 1 towards agents 2 and 3 in Example 2.3 are the same and equal to an amount of 40. One could therefore require that, after correcting for the common part of mutual liabilities, agents 2 and 3 should be treated in the same way by agent 1. Nevertheless, under the proportional rule, agent 1 pays 320 to agent 2, so less than the common part of their mutual liabilities equal to 360, and 32 to agent 3, so significantly more than the common part of their mutual liabilities, which is equal to 0. $\triangle$

The disturbing situation of Example 2.3 can be avoided by basing the payments on the net liabilities. For $i, j \in N$, let $\underline{L}_{ij} = \min\{L_{ij}, L_{ji}\}$ be the common part of the liabilities between agents i and j. Under pairwise netting, agents i and j cancel the common part of their liabilities, which in technical terms is achieved by paying $\underline{L}_{ij} = \underline{L}_{ji}$ to one another. Next, the proportional rule is applied to the net liability matrix $\widehat{L}$, where, for $i, j \in N$, $\widehat{L}_{ij} = L_{ij} - \underline{L}_{ij}$. Note that for the net liabilities, for every $i, j \in N$, $\widehat{L}_{ij} = 0$ or $\widehat{L}_{ji} = 0$. We also remark that pairwise netting between agents i and j can be viewed as giving priority to payments between one another, up to an amount of $\underline{L}_{ij}$. These considerations lead to the following definition of the pairwise netting proportional rule.

Definition 2.4. The *pairwise netting proportional rule* $b^{\mathrm{pnp}} : \mathcal{F} \to \mathcal{M}$ is defined by

$$b^{\mathrm{pnp}}(N, z, L) = \underline{L} + b^{\mathrm{p}}(N, z, \widehat{L}), \quad (N, z, L) \in \mathcal{F}.$$

Pairwise netting does not affect the initial endowments, but reduces the liabilities from L to $\widehat{L}$, so after making mutual payments according to $\underline{L}$, the resulting financial network is given by $(N, z, \widehat{L})$ and applying the proportional rule leads to additional payments $b^{\mathrm{p}}(N, z, \widehat{L})$. The pairwise netting proportional rule is well-defined as the matrix $\underline{L}$ is uniquely determined by the liability matrix L and the resulting financial network $(N, z, \widehat{L})$ with the net liability matrix belongs to $\mathcal{F}$, so $b^{\mathrm{p}}(N, z, \widehat{L})$ is well-defined as argued before.

Example 2.5 illustrates the pairwise netting proportional rule for the primitives of Example 2.3.

Example 2.5 (The pairwise netting proportional rule)**.** Consider the financial network $(N, z, L) \in \mathcal{F}$ of Example 2.3. Table 2 shows the liability matrix L, the net liability matrix $\widehat{L}$, the payment matrix $\widehat{P} = b^{\mathrm{p}}(N, z, \widehat{L})$ resulting from the proportional rule using the net liability matrix, the payment matrix $P = b^{\mathrm{pnp}}(N, z, L)$, asset values $a(N, z, P)$, and equities $e(N, z, P)$.

z	L			$\widehat{L}$			$\widehat{P}$			P			$a(N,z,P)$	$e(N,z,P)$
10	0	400	40	0	40	40	0	5	5	0	365	5	370	0
1	360	0	20	0	0	0	0	0	0	360	0	20	406	26
40	0	40	0	0	20	0	0	20	0	0	40	0	65	25

Table 2: The initial endowments, liability matrix, net liability matrix, the payment matrix based on the net liabilities, the full payment matrix, asset values, and equities in Example 2.5.

Under the pairwise netting proportional rule, agent 2 is solvent and contagion default is avoided. Agent 2 receives $360 + 20 = 380$ as a result of pairwise netting all mutual liabilities. Next, agent 2 receives 5 units from agent 1 and 20 units from agent 3 after applying the proportional rule to the net liability matrix. Agent 2 therefore ends up with an asset value of 406, which is sufficient to fully pay the liabilities of 380. In contrast, under the proportional rule, agent 2 has an asset value of 361 only and is forced to default. $\triangle$

The following proposition generalizes the insights of Examples 2.3 and 2.5 and states that an agent with higher claims than liabilities towards all other agents is always solvent under the pairwise netting proportional rule.

Proposition 2.6. *Let $F = (N, z, L) \in \mathcal{F}$ be a financial network and let $i \in N$ be such that, for every $j \in N$, $L_{ji} \geq L_{ij}$. Then it holds for $P = b^{\mathrm{pnp}}(F)$ that, for every $j \in N$, $P_{ij} = L_{ij}$, and $e_i(N, z, P) > 0$.*

Proof. Let $j \in N$. We have that $L_{ji} \geq L_{ij}$, so $\underline{L}_{ij} = L_{ij}$ and $\widehat{L}_{ij} = L_{ij} - \underline{L}_{ij} = 0$. It follows that $\widehat{P}_{ij} = b^{\mathrm{p}}_{ij}(N, z, \widehat{L}) = 0$. From the definition of b^{pnp}, we obtain that

$$P_{ij} = b^{\mathrm{pnp}}_{ij}(F) = \underline{L}_{ij} + \widehat{P}_{ij} = L_{ij} + 0 = L_{ij}.$$

Finally, it holds that

$$\begin{aligned} e_i(N, z, P) &= z_i + \textstyle\sum_{j \in N}(P_{ji} - P_{ij}) \\ &= z_i + \textstyle\sum_{j \in N}(\underline{L}_{ji} + \widehat{P}_{ji} - \underline{L}_{ij} - \widehat{P}_{ij}) \\ &= z_i + \textstyle\sum_{j \in N} \widehat{P}_{ji} \\ &\geq z_i > 0. \end{aligned}$$

□

The result of Proposition 2.6 does not depend on the application of the proportional rule to the net liabilities. What is crucial is that first one round of pairwise netting is performed and next payments are made based on the net liability matrix. Rules different from the proportional one would also lead to positive equity and solvency under the assumptions of the proposition.

As a final contribution of this section, we would like to point out that it is irrelevant whether the actual payments made are equal to $P = b^{\mathrm{pnp}}(N, z, L)$ or to $\widehat{P} = P - \underline{L} = b^{\mathrm{p}}(N, z, \widehat{L})$ as both payment matrices lead to the same values for equity.

Proposition 2.7. *Let $F = (N, z, L) \in \mathcal{F}$ be a financial network. Then it holds for $P = b^{\mathrm{pnp}}(N, z, L)$ and $\widehat{P} = b^{\mathrm{p}}(N, z, \widehat{L})$ that*

$$e(N, z, P) = e(N, z, \widehat{P}).$$

Proof. Let $i \in N$. It holds that

$$\begin{aligned} e_i(N, z, P) &= z_i + \textstyle\sum_{j \in N}(P_{ji} - P_{ij}) \\ &= z_i + \textstyle\sum_{j \in N}(P_{ji} - \underline{L}_{ji} + \underline{L}_{ij} - P_{ij}) \\ &= z_i + \textstyle\sum_{j \in N}(\widehat{P}_{ji} - \widehat{P}_{ij}) \\ &= e_i(N, z, \widehat{P}). \end{aligned}$$

□

Similar to Proposition 2.6, the result of Proposition 2.7 does not depend on the application of the proportional bankruptcy rule after pairwise netting, but would hold for other bankruptcy rules as well.

3 The Axioms

In this section, we introduce the new substantive axiom of net impartiality and we define the existing substantive axiom of invariance to mitosis. We also present the basic axioms of claims boundedness, limited liability, priority of creditors, and continuity.

In claims problems, there is a single estate and there are multiple claimants. A claims problem can be thought of as a financial network where all rows of the liability matrix but one are equal to zero. A common axiom in claims problem is impartiality, which requires that if two agents have the same claim on the estate, then they should receive the same amount. Common division rules like the proportional rule, the constrained equal awards rule, the constrained equal losses rule, and the Talmud rule satisfy impartiality. There are several natural ways in which impartiality for claims problems can be extended to an axiom for financial networks. One such extension, introduced in Csóka and Herings (2021), is presented next.

Axiom 3.1 (*Impartiality)*. For every $F = (N, z, L) \in \mathcal{F}$, for every $i, j, k \in N$ such that $L_{ij} = L_{ik}$, it holds that $b_{ij}(F) = b_{ik}(F)$.

Impartiality requires that if agent i has the same liability to agent j and agent k, then agent i should pay the same to agent j and agent k.

As argued before, two agents can decrease or increase the common part of the mutual liabilities at will. It is then natural to require that the common part of mutual liabilities is ignored when judging whether two agents are in the same position vis-à-vis a debtor. Net impartiality therefore compares the net liabilities $L_{ij} - \underline{L}_{ij}$ and $L_{ik} - \underline{L}_{ik}$. In case those are equal, then net impartiality imposes that agent i makes the same payments to agents j and k, when ignoring the payment that corresponds to the cancelation of the common part of the mutual liabilities.

Axiom 3.2 (*Net impartiality)*. For every $F = (N, z, L) \in \mathcal{F}$, for every $i, j, k \in N$ such that $L_{ij} - \underline{L}_{ij} = L_{ik} - \underline{L}_{ik}$, it holds that $b_{ij}(F) - \underline{L}_{ij} = b_{ik}(F) - \underline{L}_{ik}$.

Net impartiality requires that if agent i has the same net liability to agent j and agent k, then agent i should pay the same amounts to agents j and k after a correction for the common part of their mutual liabilities. Both impartiality and net impartiality extend the axiom of impartiality for claims problems to the setting of financial networks. This paper argues that net impartiality is a more compelling axiom in the set-up of financial networks and studies its implications.

Principles of impartiality are common in various legal systems. For instance, the EC Council Regulation on insolvency proceedings requires that creditors with the same standing should obtain the same proportion of their claims and American bankruptcy law imposes that claimants of equal status should receive payments proportional to the value of their liabilities. This principle is both applied with and without netting mutual liabilities. An example of the former situation concerns over-the-counter derivates trades, where claims are revised by one round of pairwise netting before the proportional rule is applied, see Duffie and Zhu (2011).

Net impartiality is satisfied by the pairwise netting proportional rule in Example 2.5, since agent 1 makes the same payments to agents 2 and 3 in $\widehat{P}$. However, net impartiality is not satisfied by the proportional rule in Example 2.3, since after subtracting the common part of their mutual liabilities from the respective elements of the payment matrix P, agent 1 makes a payment of -40 to agent 2 and a payment of 32 to agent 3.

Our second substantial axiom, invariance to mitosis, was introduced in Csóka and Herings (2021).

Axiom 3.3 (*Invariance to mitosis*)**.** For every $F = (N, z, L) \in \mathcal{F}$, for every $j \in N$, for every $K \subset \mathbb{N} \setminus N$, the payments in the financial network $F' = (N', z', L') \in \mathcal{F}$, where

$$\begin{array}{rcll} N' & = & N \cup K, & \\ z'_k & = & z_j/(|K|+1), & k \in \{j\} \cup K, \\ z'_i & = & z_i, & i \in N \setminus \{j\}, \\ L'_{ki} & = & L_{ji}/(|K|+1), & k \in \{j\} \cup K,\ i \in N \setminus \{j\}, \\ L'_{ik} & = & L_{ij}/(|K|+1), & k \in \{j\} \cup K,\ i \in N \setminus \{j\}, \\ L'_{k\ell} & = & 0, & k, \ell \in \{j\} \cup K, \\ L'_{hi} & = & L_{hi}, & h, i \in N \setminus \{j\}, \end{array}$$

satisfy

$$\begin{array}{rcll} \sum_{k \in \{j\} \cup K} b_{ki}(F') & = & b_{ji}(F), & i \in N \setminus \{j\}, \\ \sum_{k \in \{j\} \cup K} b_{ik}(F') & = & b_{ij}(F), & i \in N \setminus \{j\}, \\ b_{hi}(F') & = & b_{hi}(F), & h, i \in N \setminus \{j\}. \end{array}$$

The term mitosis refers to splitting an agent into multiple identical agents with the same initial endowments, claims, and liabilities. In the definition, the split of agent j into multiple identical agents is achieved by taking a set of agents K outside the financial network F and distributing all endowments, claims, and liabilities of agent j equally over the agents in $\{j\} \cup K$. Invariance to mitosis requires that such a split should not affect the payments made to and received from agents not involved in the split.

Invariance to mitosis is related to the axiom of strategy-proofness by O'Neill (1982) for the class of simple claims problems, which are claims problems where no agent has a claim exceeding the value of the estate. This axiom was used under the name non-manipulability by de Frutos (1999), Moreno-Ternero (2006), and Ju, Miyagawa, and Sakai (2007) to axiomatize the proportional rule in claims problems. Non-manipulability requires that no group of agents can increase their total awards by merging their claims and that no single agent can be better off by splitting the claim on the estate among the agent and an arbitrary number of dummy agents. Strong non-manipulability requires that such merging and splitting changes nothing for the other agents involved in the problem. This axiom was introduced by Curiel, Maschler, and Tijs (1987) as the additivity of claims property.

Invariance to mitosis only applies to cases where an agent splits into identical agents or where a group of identical agents merges into a single agent and is therefore much weaker than straightforward extensions of non-manipulability to financial networks. Indeed, as shown in Csóka and Herings (2021), there is no bankruptcy rule satisfying non-manipulability together with the basic axioms of claims boundedness, limited liability, and priority of creditors. The intuition for this result is relatively straightforward. If a bankrupt agent is allowed to create a spin-off which receives all the liabilities of the bankrupt agent, then the agent that was bankrupt before will now end up with positive equity. Such manipulations are indeed deemed illegal by actual bankruptcy law. On the contrary, the split of an agent into two identical agents occurs frequently, in particular in the form of a divorce when married in community of property.

We finally present a number of basic axioms and provide a brief interpretation of them.

Axiom 3.4 (*Claims boundedness*)**.** For every $F \in \mathcal{F}$, it holds that $b(F) \leq L$.

Claims boundedness requires that all payments should be bounded from above by the respective liabilities. It is a very natural axiom that is satisfied by almost all bankruptcy rules studied in the literature. The only exception we are aware of is the constrained-proportional rule of Demange (2022), where sometimes agents are required to pay more than their liabilities in order to rescue other agents.

Axiom 3.5 (*Limited liability*)**.** For every $F \in \mathcal{F}$, for every $i \in N$, it holds that $e_i(N, z, b(F)) \geq 0$.

Limited liability requires that all agents should end up with non-negative equity.

Axiom 3.6 (*Priority of creditors*)**.** For every $F \in \mathcal{F}$, for every $i \in N$, if $b_i(F) < L_i$, then it holds that $e_i(N, z, b(F)) = 0$.

Priority of creditors requires that an agent is only allowed to default when the agent's equity is equal to zero.

The final basic axiom is continuity. To define it, we endow $\mathcal{F}$ with the standard topology, based on the discrete topology for $\mathcal{N}$ and the Euclidean topology for initial endowments and liabilities. Let $(F^n)_{n\in\mathbb{N}} = (N^n, z^n, L^n)_{n\in\mathbb{N}}$ be a sequence of financial networks of $\mathcal{F}$. This sequence converges to the financial network $\overline{F} = (\overline{N}, \overline{z}, \overline{L})$ of $\mathcal{F}$ if and only if there is $n' \in \mathbb{N}$ such that for every $n \geq n'$ it holds that $N^n = \overline{N}$, $\lim_{n\to\infty} z^n = \overline{z}$, and $\lim_{n\to\infty} L^n = \overline{L}$.

Axiom 3.7 (*Continuity*)**.** It holds that b is continuous.

When a bankruptcy rule satisfies continuity, small changes in a financial network do not lead to large changes in payments.

4 A Characterization of the Pairwise Netting Proportional Rule

Before turning to the axiomatization of the pairwise netting proportional rule, it is helpful to characterize it in this section as the unique solution to a particular system of equations.

Let $F = (N, z, L) \in \mathcal{F}$ be a financial network. Let $\widehat{\Pi} \in \mathbb{R}_+^{N \times N}$ denote the *relative net liability matrix*, where for $i, j \in N$,

$$\widehat{\Pi}_{ij} = \begin{cases} 0, & \text{if } \widehat{L}_{ij} = 0, \\ \frac{\widehat{L}_{ij}}{\sum_{k \in N} \widehat{L}_{ik}}, & \text{otherwise.} \end{cases}$$

It follows from Definition 2.4 that $b^{\mathrm{pnp}}(F) - \underline{L} = b^{\mathrm{p}}(N, z, \widehat{L})$, so we can use Equation (2.1) and Theorem 2 of Eisenberg and Noe (2001) to characterize $b^{\mathrm{pnp}}(F) - \underline{L}$ as the unique solution to the system of equations and inequalities

$$\begin{array}{rcll} \widehat{P}_{ij} & = & \min\left\{\widehat{\Pi}_{ij}(z_i + \sum_{k \in N} \widehat{P}_{ki}), \widehat{L}_{ij}\right\}, & i, j \in N, \\ \widehat{P}_{ij} & \geq & 0, & i, j \in N. \end{array} \tag{4.2}$$

The matrix $\widehat{P}$ corresponds to the payments in excess of the common part of the liabilities $\underline{L}$. For a general payment matrix there is no reason that the payments in excess of the liabilities are non-negative. Such non-negativity is also not directly implied by the axioms that we have made. We therefore relax, for every $i, j \in N$, the inequality constraint $\widehat{P}_{ij} \geq 0$ to $\widehat{P}_{ij} \geq -\underline{L}_{ij}$, and obtain the system of equations and inequalities

$$\begin{array}{rcll} \widehat{P}_{ij} & = & \min\left\{\widehat{\Pi}_{ij}(z_i + \sum_{k \in N} \widehat{P}_{ki}), \widehat{L}_{ij}\right\}, & i, j \in N, \\ \widehat{P}_{ij} & \geq & -\underline{L}_{ij}, & i, j \in N, \end{array} \tag{4.3}$$

The crucial point of this section is to show that the relaxed system of equations and inequalities (4.3) remains to have a unique solution, which must therefore be equal to $b^{\mathrm{pnp}}(F) - \underline{L}$.

There is an extensive literature on the uniqueness of clearing payment matrices in financial networks, see Eisenberg and Noe (2001), Glasserman and Young (2015), Groote Schaarsberg, Reijnierse, and Borm (2018), and Koster (2019). Csóka and Herings (2023) provides sufficient conditions for uniqueness that weaken all previous conditions provided in the literature. However, all these papers rely on non-negativity constraints as in (4.2), so do not imply uniqueness of the solution to (4.3).

To show the uniqueness of the solution to (4.3), we need some preliminary lemmas. Our first lemma states that the set of solutions to the system of equations and inequalities in (4.3) is a complete lattice with $b^{\mathrm{pnp}}(F) - \underline{L}$ as the greatest solution. It also establishes that every solution leads to the same values for equity.

Lemma 4.1. *Let $F = (N, z, L) \in \mathcal{F}$ be a financial network. The set of solutions to the system of equations and inequalities (4.3) is a complete lattice. The greatest solution $\widehat{P}^{+}$ is equal to $b^{\mathrm{pnp}}(F) - \underline{L}$. For every solution $\widehat{P}^{*}$ to (4.3), it holds that $e(N, z, \widehat{P}^{*}) = e(N, z, \widehat{P}^{+}) \geq 0$. Moreover, for every $i \in N$, if $e_i(N, z, \widehat{P}^{*}) = 0$ then $\widehat{P}^{*}_{i} = \widehat{\Pi}_i(z_i + \sum_{j \in N} \widehat{P}^{*}_{ji})$ and if $e_i(N, z, \widehat{P}^{*}) > 0$ then $\widehat{P}^{*}_{i} = \widehat{L}_i$.*

Proof. The proof proceeds in four steps. Step 1 introduces a function φ such that any solution to (4.3) is a fixed point of φ. Step 2 derives that the set of fixed points of φ is a complete lattice. In Step 3 it is shown that every fixed point results in the same equity for all agents and that a solution with positive equity for some agent implies that the corresponding row of the payment matrix is equal to the net liability of that agent. Step 4 derives that every fixed point of φ is a solution to (4.3) and that the solution satisfies the further properties as stated in the lemma.

Step 1. *Construction of the function φ.*

The set of feasible net payment matrices is defined by

$$\widehat{\mathcal{P}} = \{\widehat{P} \in \mathbb{R}^{N \times N} \mid -\underline{L} \leq \widehat{P} \leq \widehat{L}\}.$$

Let $\varphi : \widehat{\mathcal{P}} \to \widehat{\mathcal{P}}$ be defined by

$$\varphi_{ij}(\widehat{P}) = \max\left\{-\underline{L}_{ij}, \min\left\{\widehat{\Pi}_{ij}(z_i + \textstyle\sum_{k \in N} \widehat{P}_{ki}), \widehat{L}_{ij}\right\}\right\}, \quad \widehat{P} \in \widehat{\mathcal{P}},\ i, j \in N.$$

Let $\widehat{P}^{*}$ be a solution to the system of equations and inequalities (4.3). For every $i, j \in N$, it holds that

$$\begin{aligned}\varphi_{ij}(\widehat{P}^{*}) &= \max\left\{-\underline{L}_{ij}, \min\left\{\widehat{\Pi}_{ij}(z_i + \textstyle\sum_{k \in N} \widehat{P}^{*}_{ki}), \widehat{L}_{ij}\right\}\right\} \\ &= \max\left\{-\underline{L}_{ij}, \widehat{P}^{*}_{ij}\right\} \\ &= \widehat{P}^{*}_{ij},\end{aligned}$$

where the second equality follows from the first line in (4.3) and the third equality from the second line in (4.3). We have shown that $\widehat{P}^{*}$ is a fixed point of φ.

Step 2. *The set of fixed points of φ is a complete lattice. The greatest fixed point of φ is equal to $b^{\mathrm{pnp}}(F) - \underline{L}$.*

The set $\widehat{\mathcal{P}}$ is partially ordered by $\leq$. All subsets of $\widehat{\mathcal{P}}$ have both a supremum and an infimum in $\widehat{\mathcal{P}}$, so $\widehat{\mathcal{P}}$ is a complete lattice.

The function φ is monotonic, since $\widehat{\Pi}_{ij}(z_i + \sum_{k\in N} \widehat{P}_{ki})$ is monotonic and the operators min and max preserve monotonicity. By Tarski's fixed point theorem (Tarski, 1955), the set of fixed points of φ is a complete lattice with respect to $\leq$. It follows that the set of fixed points of φ has a least and a greatest element. Let $\widehat{P}^-$ and $\widehat{P}^+$ denote the least and the greatest fixed point of φ, respectively. As argued before, $b^{\mathrm{pnp}}(F) - \underline{L}$ is the unique non-negative solution to (4.3). By Step 1, $b^{\mathrm{pnp}}(F) - \underline{L}$ is a fixed point of φ. Then we have that $\widehat{P}^+ \geq b^{\mathrm{pnp}}(F) - \underline{L} \geq 0$. Since $\widehat{P}^+ \geq 0$, we have, for every $i, j \in N$, $\widehat{\Pi}_{ij}(z_i + \sum_{k\in N} \widehat{P}^+_{ki}) \geq 0$, so

$$\begin{aligned}
\widehat{P}^+_{ij} &= \varphi_{ij}(\widehat{P}^+) = \max\left\{-\underline{L}_{ij}, \min\left\{\widehat{\Pi}_{ij}(z_i + \textstyle\sum_{k\in N} \widehat{P}^+_{ki}), \widehat{L}_{ij}\right\}\right\} \\
&= \min\left\{\widehat{\Pi}_{ij}(z_i + \textstyle\sum_{k\in N} \widehat{P}^+_{ki}), \widehat{L}_{ij}\right\},
\end{aligned}$$

so $\widehat{P}^+$ is a non-negative solution to (4.3). Since $b^{\mathrm{pnp}}(F) - \underline{L}$ is the unique non-negative solution to (4.3), it follows that $\widehat{P}^+ = b^{\mathrm{pnp}}(F) - \underline{L}$.

Step 3. *For every fixed point $\widehat{P}^*$ of φ, we have that $e(N, z, \widehat{P}^*) = e(N, z, \widehat{P}^+) \geq 0$. Moreover, for every $i \in N$, if $e_i(N, z, \widehat{P}^*) > 0$ then $\widehat{P}^*_i = \widehat{L}_i$.*

Let $\widehat{P}^*$ be a fixed point of φ. We show that $e(N, z, \widehat{P}^*) \leq e(N, x, \widehat{P}^+)$, which, together with $\sum_{i\in N} e(N, z, \widehat{P}^*) = \sum_{i\in N} e(N, z, \widehat{P}^+) = \sum_{i\in N} z_i$ implies that $e(N, z, \widehat{P}^*) = e(N, z, \widehat{P}^+)$. By Step 2, $\widehat{P}^+ = b^{\mathrm{pnp}}(F) - \underline{L} = b^{\mathrm{p}}(N, z, \widehat{L})$, so $e(N, z, \widehat{P}^+) \geq 0$ follows from the fact that the proportional rule satisfies limited liability. Therefore, if for some $i \in N$ it holds that $e_i(N, z, \widehat{P}^*) \leq 0$, then $e_i(N, z, \widehat{P}^*) \leq e_i(N, z, \widehat{P}^+)$.

Let $i \in N$ be such that $e_i(N, z, \widehat{P}^*) > 0$. We show that, for every $j \in N$, $\widehat{P}^*_{ij} = \widehat{L}_{ij}$. If, for some $j \in N$, $\widehat{L}_{ij} = 0$, then we have $\widehat{\Pi}_{ij} = 0$, so

$$\begin{aligned}
\widehat{P}^*_{ij} &= \varphi_{ij}(\widehat{P}^*) \\
&= \max\{-\underline{L}_{ij}, \min\{\widehat{\Pi}_{ij}(z_i + \textstyle\sum_{k\in N} \widehat{P}^*_{ki}), 0\}\} \\
&= 0.
\end{aligned}$$

It follows that $\widehat{L}_i = 0$ implies $\widehat{P}^*_i = 0$.

Consider the case $\widehat{L}_i > 0$. Since

$$\begin{aligned}
e_i(N, z, \widehat{P}^*) &= z_i + \textstyle\sum_{k\in N} \widehat{P}^*_{ki} - \sum_{j\in N} \widehat{P}^*_{ij} \\
&= \textstyle\sum_{j\in N} \widehat{\Pi}_{ij}(z_i + \sum_{k\in N} \widehat{P}^*_{ki}) - \sum_{j\in N} \widehat{P}^*_{ij} > 0,
\end{aligned}$$

it follows that there is $j' \in N$ such that

$$\widehat{P}^*_{ij'} < \widehat{\Pi}_{ij'}(z_i + \textstyle\sum_{k\in N} \widehat{P}^*_{ki}). \tag{4.4}$$

It also holds that $L_{ij'} > 0$ since otherwise both sides of inequality (4.4) are equal to zero. From (4.4) and $\widehat{P}^*_{ij'} = \varphi_{ij'}(\widehat{P}^*)$ it follows that

$$\widehat{P}^*_{ij'} = \widehat{L}_{ij'} < \widehat{\Pi}_{ij'}(z_i + \textstyle\sum_{k\in N}\widehat{P}^*_{ki}), \tag{4.5}$$

so $z_i + \sum_{k\in N}\widehat{P}^*_{ki} > 0$.

For every $j \in N$, multiplying (4.5) by $\widehat{L}_{ij}/\widehat{L}_{ij'}$, we get that

$$\min\left\{\widehat{\Pi}_{ij}(z_i + \textstyle\sum_{k\in N}\widehat{P}^*_{ki}), \widehat{L}_{ij}\right\} \;=\; \widehat{L}_{ij} \geq -\underline{L}_{ij},$$

implying that $\widehat{P}^*_{ij} = \widehat{L}_{ij}$.

Having established that $\widehat{P}^*_i = \widehat{L}_i$, we find that

$$\begin{aligned} e_i(N, z, \widehat{P}^*) &= z_i + \textstyle\sum_{j\in N}\widehat{P}^*_{ji} - \sum_{j\in N}\widehat{L}_{ij} \\ &\leq z_i + \textstyle\sum_{j\in N}\widehat{P}^+_{ji} - \sum_{j\in N}\widehat{L}_{ij} \\ &= e_i(N, z, \widehat{P}^+), \end{aligned}$$

where the last equality uses that $\widehat{L}_{ij} \geq \widehat{P}^+_{ij} \geq P^*_{ij} = \widehat{L}_{ij}$.

Step 4. *Every fixed point $\widehat{P}^*$ of φ is a solution to* (4.3). *Moreover, for every $i \in N$, if $e_i(N, z, \widehat{P}^*) = 0$ then $\widehat{P}^*_i = \widehat{\Pi}_i(z_i + \sum_{j\in N}\widehat{P}^*_{ji})$.*

Let $\widehat{P}^*$ be a fixed point of φ. Suppose, by contradiction, that there is i, j' such that

$$\widehat{P}^*_{ij'} = -\underline{L}_{ij'} > \widehat{\Pi}_{ij'}(z_i + \textstyle\sum_{k\in N}\widehat{P}^*_{ki}). \tag{4.6}$$

Then it should hold that, for every $j \in N$,

$$\widehat{P}^*_{ij} \geq \widehat{\Pi}_{ij}(z_i + \textstyle\sum_{k\in N}\widehat{P}^*_{ki}) \tag{4.7}$$

since otherwise there is a $k' \in N$ such that $\widehat{P}^*_{ik'} = \widehat{L}_{ik'} < \widehat{\Pi}_{ik'}(z_i + \sum_{k\in N}\widehat{P}^*_{ki})$, implying that $z_i + \sum_{k\in N}\widehat{P}^*_{ki} > 0$, contradicting (4.6). Equations (4.6) and (4.7) imply that $e_i(N, z, \widehat{P}^*) < 0$, a contradiction to Step 3. Consequently, every fixed point $\widehat{P}^*$ of φ is a solution to (4.3).

Let $\widehat{P}^*$ be a solution to (4.3) such that, for some $i \in N$, $e_i(N, z, \widehat{P}^*) = 0$. Since $\widehat{L}_i = 0$ implies $\widehat{P}^+_i = 0$, so $e_i(N, z, \widehat{P}^*) = e_i(N, z, \widehat{P}^+) > 0$, we have that $\widehat{L}_i > 0$. It holds that

$$\begin{aligned} \textstyle\sum_{j\in N}\min\{\widehat{\Pi}_{ij}(z_i + \sum_{k\in N}\widehat{P}^*_{ki}), \widehat{L}_{ij}\} &= \textstyle\sum_{j\in N}\widehat{P}^*_{ij} \\ &= z_i + \textstyle\sum_{k\in N}\widehat{P}^*_{ki} \\ &= \textstyle\sum_{j\in N}\widehat{\Pi}_{ij}(z_i + \sum_{k\in N}\widehat{P}^*_{ki}), \end{aligned}$$

where the first equality follows from (4.3), the second from $e_i(N, z, \widehat{P}^*) = 0$, and the third since $\sum_{j\in N}\widehat{\Pi}_{ij} = 1$. It follows that, for every $j \in N$, $\widehat{P}^*_{ij} = \widehat{\Pi}_{ij}(z_i + \sum_{k\in N}\widehat{P}^*_{ki})$. □

The proof of Lemma 4.1 is based on the construction of a function that satisfies the assumptions of Tarski's fixed point theorem and which is such that any solution to (4.3) yields a fixed point of the function. It follows that the set of fixed points is a complete lattice. It is easy to show that $b^{\text{pnp}}(F) - \underline{L}$ must be the greatest fixed point. After deriving a number of useful properties of the fixed points, the proof of the lemma is completed by showing that every fixed point yields a solution to (4.3).

We now need to have a closer look at the properties of a solution to (4.3). We use the following definitions from Csóka and Herings (2023). A sequence of $k' \geq 2$ distinct agents $(i_1, \ldots, i_{k'})$ is a *directed path* in a matrix $M \in \mathcal{M}(N)$ if, for every $k \in \{1, \ldots, k'-1\}$, $M_{i_k i_{k+1}} > 0$. Agent $j \in N$ is connected to agent $i \in N$ in M if there is a directed path $(i_1, \ldots, i_{k'})$ in M such that $i_1 = i$ and $i_{k'} = j$.

Let $F = (N, z, L) \in \mathcal{F}$ be a financial network. A set of agents $S \subset N$ is said to be a *strongly connected component* in L if any two distinct agents in S are connected to each other in L and the set S is maximal with regard to this property.

For every $i \in N$, let $O(i)$ denote the strongly connected component in L to which i belongs. The collection $\mathcal{O} = \{O(i) \mid i \in N\}$ is a partition of N. The directed graph $(\mathcal{O}, D)$ is defined by

$$D = \{(O, O') \in \mathcal{O} \times \mathcal{O} \mid \exists i \in O,\ \exists j \in O',\ L_{ij} > 0\},$$

so, for two distinct elements $O, O' \in \mathcal{O}$, there is an arc from O to O' if there is $i \in O$ and $j \in O'$ such that $L_{ij} > 0$. The successors of $O \in \mathcal{O}$ in the directed graph $(\mathcal{O}, D)$ are given by the strongly connected components that are connected to O in $(\mathcal{O}, D)$. The directed graph $(\mathcal{O}, D)$ has no cycles. We can therefore order the sets in $\mathcal{O}$ and write $\mathcal{O} = \{O_1, \ldots, O_R\}$, where $(O_r, O_{r'}) \in D$ implies $r < r'$. In general, this order is not uniquely determined, for instance when $\mathcal{O}$ has at least two elements and at least one strongly connected component $O \in \mathcal{O}$ has neither predecessors nor successors.

An agent $i \in N$ is said to be a *cyclical agent* and the set $O(i)$ is said to be a *cycle* if $O(i)$ consists of at least two elements. Agent i is cyclical if and only if there is a directed path of agents in L starting at agent i such that the last agent on the path has a positive liability towards agent i.

We denote the least solution to (4.3) by $\widehat{P}^-$. Let $\widehat{A}^-, \widehat{A}^+ \in \mathbb{R}^N$ be the vectors of asset values defined by

$$\begin{array}{rcll} \widehat{A}_i^- & = & z_i + \sum_{k \in N} \widehat{P}_{ki}^-, & \quad i \in N, \\ \widehat{A}_i^+ & = & z_i + \sum_{k \in N} \widehat{P}_{ki}^+, & \quad i \in N. \end{array}$$

The next two lemmas are used to show that, for every $i \in N$, $\widehat{A}_i^- = \widehat{A}_i^+$.

The first lemma considers the case when all agents in a cycle end up with zero equity at a solution to (4.3). If there is an agent i in the cycle such that $\widehat{A}_i^- < \widehat{A}_i^+$, then this holds for all agents in the cycle.

Lemma 4.2. *Let $F = (N, z, L) \in \mathcal{F}$ be a financial network. Let $O \in \mathcal{O}$ be a cycle such that $\sum_{i \in O} e_i(N, z, \widehat{P}^+) = 0$. If there is an agent $i^1 \in O$ such that $\widehat{A}_{i^1}^- < \widehat{A}_{i^1}^+$, then, for every $i \in O$, $\widehat{A}_i^- < \widehat{A}_i^+$.*

Proof. Let $i \in O$. It holds that $e_i(\widehat{P}^-) = e_i(\widehat{P}^+) = 0$. It follows from Lemma 4.1 that $\widehat{P}_i^- = \widehat{\Pi}_i \widehat{A}_i^-$ and $\widehat{P}_i^+ = \widehat{\Pi}_i \widehat{A}_i^+$.

Let $i^1 \in O$ be such that $\widehat{A}_{i^1}^- < \widehat{A}_{i^1}^+$. Let $i^2 \in O \setminus \{i^1\}$. Let $j^1, \ldots, j^m \in O$ be such that $j^1 = i^1, j^m = i^2$, and for $\ell = 1, \ldots, m-1$, we have that $\widehat{L}_{j^\ell j^{\ell+1}} > 0$.

By assumption it holds that $\widehat{A}_{j^1}^- < \widehat{A}_{j^1}^+$. Let us assume, for some $\ell = 1, \ldots, m-1$, that $\widehat{A}_{j^\ell}^- < \widehat{A}_{j^\ell}^+$. We show that $\widehat{A}_{j^{\ell+1}}^- < \widehat{A}_{j^{\ell+1}}^+$. We have that

$$\widehat{A}_{j^{\ell+1}}^- = z_{j^{\ell+1}} + \sum_{i \in N} \widehat{P}_{ij^{\ell+1}}^- < z_{j^{\ell+1}} + \sum_{i \in N} \widehat{P}_{ij^{\ell+1}}^+ = \widehat{A}_{j^{\ell+1}}^+, \tag{4.8}$$

since, for every $i \in N$, $\widehat{P}_{ij^{\ell+1}}^- \leq \widehat{P}_{ij^{\ell+1}}^+$ and $\widehat{P}_{j^\ell j^{\ell+1}}^- = \widehat{\Pi}_{j^\ell j^{\ell+1}} \widehat{A}_{j^\ell}^- < \widehat{\Pi}_{j^\ell j^{\ell+1}} \widehat{A}_{j^\ell}^+ = \widehat{P}_{j^\ell j^{\ell+1}}^+$, because $\widehat{\Pi}_{j^\ell j^{\ell+1}} > 0$ and $\widehat{A}_{j^\ell}^- < \widehat{A}_{j^\ell}^+$. □

The next lemma considers the case when there is at least one agent with positive equity in a cycle. In that case, all agents in the cycle have the same asset value at all solutions.

Lemma 4.3. *Let $F = (N, z, L) \in \mathcal{F}$ be a financial network. Let $O \in \mathcal{O}$ be a cycle such that $\sum_{i \in O} e_i(N, z, \widehat{P}^+) > 0$. Then we have that, for every $i \in O$, $\widehat{A}_i^- = \widehat{A}_i^+$.*

Proof. Clearly, there is an agent $i^0 \in O$ such that $e_{i^0}(N, z, \widehat{P}^+) > 0$.

Suppose $i^1 \in O$ is such that $\widehat{A}_{i^1}^- < \widehat{A}_{i^1}^+$.

Let $j^1, \ldots, j^{m'} \in O$ be such that $j^1 = i^1, j^{m'} = i^0$, and for $\ell = 1, \ldots, m'-1$, we have that $\widehat{L}_{j^\ell j^{\ell+1}} > 0$. Let $m \in \{1, \ldots, m'\}$ be such that $e_{j^m}(\widehat{P}^+) > 0$ and for $\ell < m$ we have that $e_{j^\ell}(\widehat{P}^+) = 0$. Notice that m can be equal to 1.

We show by induction that, for every $\ell = 1 \ldots, m$, $\widehat{A}_{j^\ell}^- < \widehat{A}_{j^\ell}^+$.

For $\ell = 1$, it holds that $\widehat{A}_{j^1}^- = \widehat{A}_{i^1}^- < \widehat{A}_{i^1}^+ = \widehat{A}_{j^1}^+$.

For some $\ell < m$, assume that $\widehat{A}_{j^\ell}^- < \widehat{A}_{j^\ell}^+$. Since $e_{j^\ell}(N, z, \widehat{P}^-) = e_{j^\ell}(N, z, \widehat{P}^+) = 0$, it holds by Lemma 4.1 that $\widehat{P}_{j^\ell}^- = \widehat{\Pi}_{j^\ell} \widehat{A}_{j^\ell}^-$ and $\widehat{P}_{j^\ell}^+ = \widehat{\Pi}_{j^\ell} \widehat{A}_{j^\ell}^+$. Then we have that

$$\widehat{A}_{j^{\ell+1}}^- = z_{j^{\ell+1}} + \sum_{i \in N} \widehat{P}_{ij^{\ell+1}}^- < z_{j^{\ell+1}} + \sum_{i \in N} \widehat{P}_{ij^{\ell+1}}^+ = \widehat{A}_{j^{\ell+1}}^+. \tag{4.9}$$

Now for the solvent agent j^m, using the inequality in (4.9) and $\widehat{P}_{j^m} = \widehat{L}_{j^m}$ as shown in Lemma 4.1, we get

$$e_{j^m}(N, z, \widehat{P}^-) = A_{j^m}^- - \sum_{i \in N} \widehat{L}_{j^m i} < A_{j^m}^+ - \sum_{i \in N} \widehat{L}_{j^m i} = e_{j^m}(\widehat{P}^+), \tag{4.10}$$

contradicting the statement of Lemma 4.1. Consequently, for every $i^1 \in O$ it holds that $\widehat{A}^-_{i^1} = \widehat{A}^+_{i^1}$. $\square$

The following proposition states the main result of this section. For every $F \in \mathcal{F}$, the payment matrix $b^{\mathrm{pnp}}(F)$ resulting from the pairwise netting proportional rule can be characterized as the unique solution of the system of equations and inequalities (4.3).

Proposition 4.4. *Let $F = (N, z, L) \in \mathcal{F}$ be a financial network. The system of equations and inequalities (4.3) has $b^{\mathrm{pnp}}(F) - \underline{L}$ as its unique solution.*

Proof. Let $\mathcal{O} = \{O_1, \ldots, O_R\}$ be the partition of strongly connected components of agents determined by L, where $(O_r, O_{r'}) \in D$ implies $r < r'$. First, we show that, for every $i \in O_1$, $\widehat{A}^-_i = \widehat{A}^+_i$. We distinguish two cases.

Case 1. O_1 is a singleton.

Let $i \in N$ be such that $O_1 = \{i\}$. It clearly holds that $\widehat{A}^-_i = \widehat{A}^+_i = z_i$.

Case 2. O_1 is a cycle.

Subcase 2a. $\sum_{i \in O_1} e_i(N, z, \widehat{P}^+) > 0$.

By Lemma 4.3 we have that, for every $i \in O_1$, $\widehat{A}^-_i = \widehat{A}^+_i$.

Subcase 2b. $\sum_{i \in O_1} e_i(N, z, \widehat{P}^+) = 0$.

Since, for every $i \in O_1$, $z_i > 0$, there is $i' \in O_1$ such that $\widehat{A}^+_{i'} > 0$ and there is $j \notin O_1$ such that $\widehat{P}^+_{i'j} > 0$.

Suppose that $\widehat{A}^-_{i'} < \widehat{A}^+_{i'}$. Then, using that at $\widehat{P}^+$ equity is zero for all agents, it follows that $\widehat{P}^-_{i'j} < \widehat{P}^+_{i'j}$, and we have that

$$\sum_{i \in O_1} e_i(N, z, \widehat{P}^-) = \sum_{i \in O_1} (z_i - \sum_{k \in N \setminus O_1} \widehat{P}^-_{ik}) > \sum_{i \in O_1} (z_i - \sum_{k \in N \setminus O_1} \widehat{P}^+_{ik}) = \sum_{i \in O_1} e_i(N, z, \widehat{P}^+), \quad (4.11)$$

contradicting Lemma 4.1. Consequently, we have that $\widehat{A}^-_{i'} = \widehat{A}^+_{i'}$, so by Lemma 4.2, for every $i \in O_1$, $\widehat{A}^-_i = \widehat{A}^+_i$.

Now we continue by induction. Let us assume that for some $q' < R$, for every $q \leq q'$, for every $i \in O_q$, we have that $\widehat{A}^-_i = \widehat{A}^+_i$. We show that for every $i \in O_{q'+1}$, $\widehat{A}^-_i = \widehat{A}^+_i$.

We distinguish two cases.

Case 1. $O_{q'+1}$ is a singleton.

Let $j \in N$ be such that $O_{q'+1} = \{j\}$. We have that

$$\widehat{A}^-_j = z_j + \sum_{q=1}^{q'} \sum_{i \in O_q} \widehat{P}^-_{ij} = z_j + \sum_{q=1}^{q'} \sum_{i \in O_q} \widehat{P}^+_{ij} = \widehat{A}^+_j. \quad (4.12)$$

Case 2. $O_{q'+1}$ is a cycle.

Subcase 2a. $\sum_{i \in O_{q'+1}} e_i(N, z, \widehat{P}^+) > 0$.

By Lemma 4.3 we have that, for every $i \in O_{q'+1}$, $\widehat{A}_i^- = \widehat{A}_i^+$.

Subcase 2b. $\sum_{i \in O_{q'+1}} e_i(N, z, \widehat{P}^+) = 0$.

Since for the total asset value in $O_{q'+1}$ we have that

$$\sum_{i \in O_{q'+1}} \sum_{q=1}^{q'} \sum_{j \in O_q} \widehat{P}_{ji}^+ + \sum_{i \in O_{q'+1}} z_i > 0,$$

there is $i' \in O_{q'+1}$ such that $\widehat{A}_{i'}^+ > 0$ and $j \notin O_{q'+1}$ such that $\widehat{P}_{i'j}^+ > 0$.

Suppose that $\widehat{A}_{i'}^- < \widehat{A}_{i'}^+$. By Lemma 4.1, $\widehat{P}_i^- = \widehat{\Pi}_i \widehat{A}_i^-$ and $\widehat{P}_i^+ = \widehat{\Pi}_i \widehat{A}_i^+$, so it follows that $\widehat{P}_{i'j}^- < \widehat{P}_{i'j}^+$. We have that

$$\begin{array}{rcl} \sum_{i \in O_{q'+1}} e_i(\widehat{P}^-) & = & \sum_{i \in O_{q'+1}} (z_i + \sum_{q=1}^{q'} \sum_{j \in O_q} \widehat{P}_{ji}^-) - \sum_{i \in O_{q'+1}} \sum_{k \in N \setminus O_{q'+1}} \widehat{P}_{ik}^- \\ & > & \sum_{i \in O_{q'+1}} (z_i + \sum_{q=1}^{q'} \sum_{j \in O_q} \widehat{P}_{ji}^+) - \sum_{i \in O_{q'+1}} \sum_{k \in N \setminus O_{q'+1}} \widehat{P}_{ik}^+ \\ & = & \sum_{i \in O_{q'+1}} e_i(\widehat{P}^+), \end{array}$$

contradicting Lemma 4.1. Consequently, we have that $\widehat{A}_{i'}^- = \widehat{A}_{i'}^+$ and, by Lemma 4.2, for every $i \in O_{q'+1}$, $\widehat{A}_i^- = \widehat{A}_i^+$. $\square$

5 The Axiomatization of the Pairwise Netting Proportional Rule

This section contains the main result of the paper, stated as Theorem 5.2. The pairwise netting proportional rule satisfies the two substantive axioms of net impartiality and invariance to mitosis, as well as the basic axioms of claims boundedness, limited liability, priority of creditors, and continuity, and it is the only bankruptcy rule to do so.

The first step towards the axiomatization of the pairwise netting proportional rule is provided by the following lemma, where one net liability of an agent is a positive rational multiple of another net liability, and net impartiality and invariance to mitosis imply that payments, when corrected for the common part of mutual liabilities, are in accordance with the same positive multiple.

Lemma 5.1. *Let $F = (N, z, L) \in \mathcal{F}$ be a financial network and let $i, j, k \in N$ and $q, r \in \mathbb{N}$ be such that $\widehat{L}_{ij} = (q/r)\widehat{L}_{ik}$. Let b be a bankruptcy rule satisfying net impartiality and invariance to mitosis. Then we have $b_{ij}(F) - \underline{L}_{ij} = (q/r)(b_{ik}(F) - \underline{L}_{ik})$.*

Proof. We first consider the case $r = 1$. Let $F' = (N', z', L') \in \mathcal{F}$ be the financial network where agent j is split into q identical agents j and $\ell_1, \ldots, \ell_{q-1} \in \mathbb{N} \setminus N$, more precisely

$$
\begin{array}{rcll}
N' & = & N \cup \{\ell_1, \ldots, \ell_{q-1}\}, & \\
z'_j & = & z'_{\ell_1} = \cdots = z'_{\ell_{q-1}} = z_j/q, & \\
z'_h & = & z_h, & h \in N \setminus \{j\}, \\
L'_{jh} & = & L'_{\ell_1 h} = \cdots = L'_{\ell_{q-1} h} = L_{jh}/q, & h \in N \setminus \{j\}, \\
L'_{hj} & = & L'_{h\ell_1} = \cdots = L'_{h\ell_{q-1}} = L_{hj}/q, & h \in N \setminus \{j\}, \\
L'_{hh'} & = & 0, & h, h' \in \{j, \ell_1, \ldots, \ell_{q-1}\}, \\
L'_{hh'} & = & L_{hh'}, & h, h' \in N \setminus \{j\}.
\end{array}
\tag{5.13}
$$

Then, for the common part of the mutual liabilities, we have that

$$
\underline{L}_{ij} = \underline{L}'_{ij} + \underline{L}'_{i\ell_1} + \cdots + \underline{L}'_{i\ell_{q-1}}, \tag{5.14}
$$

since for $h \in \{j, \ell_1, \ldots, \ell_{q-1}\}$ it holds that

$$
\underline{L}'_{ih} = \min\{L'_{ih}, L'_{hi}\} = \min\{\tfrac{L_{ij}}{q}, \tfrac{L_{ji}}{q}\} = \tfrac{1}{q}\min\{L_{ij}, L_{ji}\} = \tfrac{1}{q}\underline{L}_{ij}, \tag{5.15}
$$

where the second equality follows from lines 4 and 5 in (5.13).

By invariance to mitosis, we have that

$$
b_{ij}(F) = b_{ij}(F') + b_{i\ell_1}(F') + \cdots + b_{i\ell_{q-1}}(F'),
$$

which, together with (5.14), implies that

$$
b_{ij}(F) - \underline{L}_{ij} = b_{ij}(F') - \underline{L}'_{ij} + b_{i\ell_1}(F') - \underline{L}'_{i\ell_1} + \cdots + b_{i\ell_{q-1}}(F') - \underline{L}'_{i\ell_{q-1}}. \tag{5.16}
$$

For $h \in \{j, \ell_1, \ldots, \ell_{q-1}\}$, we have that

$$
\widehat{L}'_{ih} = \widehat{L}'_{ik}, \tag{5.17}
$$

since

$$
\widehat{L}'_{ih} \;=\; L'_{ih} - \underline{L}'_{ih} = \tfrac{1}{q}L_{ij} - \tfrac{1}{q}\underline{L}_{ij} = \tfrac{1}{q}\widehat{L}_{ij} = \widehat{L}_{ik} = \widehat{L}'_{ik},
$$

where the second equality follows from (5.15), the fourth equality by the assumption of the lemma, and the last equality by the last line of (5.13). Now net impartiality together with (5.17) implies that $b_{ih}(F') - \underline{L}'_{ih} = b_{ik}(F') - \underline{L}'_{ik}$, so (5.16) becomes

$$
b_{ij}(F) - \underline{L}_{ij} = q(b_{ik}(F') - \underline{L}'_{ik}) = q(b_{ik}(F) - \underline{L}_{ik}),
$$

where the second equality follows by invariance to mitosis and the last line of (5.13).

We next consider the general case. Without loss of generality, we assume $q < r$. Let $F' = (N', z', L') \in \mathcal{F}$ be the financial network where agent k is split into r identical agents k and $\ell_1, \ldots, \ell_{r-1} \in \mathbb{N} \setminus N$, more precisely

$$\begin{array}{rcll} N' & = & N \cup \{\ell_1, \ldots, \ell_{r-1}\}, & \\ z'_k & = & z'_{\ell_1} = \cdots = z'_{\ell_{r-1}} = z_k/r, & \\ z'_h & = & z_h, & h \in N \setminus \{k\}, \\ L'_{kh} & = & L'_{\ell_1 h} = \cdots = L'_{\ell_{r-1} h} = L_{kh}/r, & h \in N \setminus \{k\}, \\ L'_{hk} & = & L'_{h\ell_1} = \cdots = L'_{h\ell_{r-1}} = L_{hk}/r, & h \in N \setminus \{k\}, \\ L'_{hh'} & = & 0, & h, h' \in \{k, \ell_1, \ldots, \ell_{r-1}\}, \\ L'_{hh'} & = & L_{hh'}, & h, h' \in N \setminus \{k\}. \end{array} \tag{5.18}$$

By invariance to mitosis, we have that

$$b_{ik}(F) = b_{ik}(F') + b_{i\ell_1}(F') + \cdots + b_{i\ell_{r-1}}(F'). \tag{5.19}$$

Since $\underline{L}_{ik} = \underline{L}'_{ik} + \underline{L}'_{i\ell_1} + \cdots + \underline{L}'_{i\ell_{r-1}}$, (5.19) becomes

$$\begin{array}{rcl} b_{ik}(F) - \underline{L}_{ik} & = & b_{ik}(F') - \underline{L}'_{ik} + b_{i\ell_1}(F') - \underline{L}'_{i\ell_1} + \cdots + b_{i\ell_{r-1}}(F') - \underline{L}'_{i\ell_{r-1}} \\ & = & r(b_{ik}(F') - \underline{L}'_{ik}) \\ & = & \frac{r}{q}(b_{ij}(F') - \underline{L}'_{ij}) \\ & = & \frac{r}{q}(b_{ij}(F) - \underline{L}_{ij}), \end{array} \tag{5.20}$$

where the second equality of (5.20) follows by net impartiality, the third equality by the first step in the proof since $L'_{ij} = qL'_{ik}$, and the fourth equality by invariance to mitosis. □

Our main theorem is the following.

Theorem 5.2. *The pairwise netting proportional rule b^{pnp} is the only bankruptcy rule satisfying the axioms of net impartiality, invariance to mitosis, claims boundedness, limited liability, priority of creditors, and continuity.*

Proof.

($\Rightarrow$) We show that b^{pnp} satisfies net impartiality. Let $F = (N, z, L) \in \mathcal{F}$, let $i, j, k \in N$ be such that $L_{ij} - \underline{L}_{ij} = L_{ik} - \underline{L}_{ik}$, and let $\widehat{P} = b^{\mathrm{p}}(N, z, \widehat{L})$. It holds that

$$\begin{array}{rcl} b^{\mathrm{pnp}}_{ij}(F) - \underline{L}_{ij} & = & \underline{L}_{ij} + b^{\mathrm{p}}_{ij}(N, z, \widehat{L}) - \underline{L}_{ij} \\ & = & \widehat{\Pi}_{ij}(z_i + \sum_{h \in N} \widehat{P}_{hi}, \widehat{L}_{ij}) \\ & = & \widehat{\Pi}_{ik}(z_i + \sum_{h \in N} \widehat{P}_{hi}, \widehat{L}_{ik}) \\ & = & \underline{L}_{ik} + b^{\mathrm{p}}_{ik}(N, z, \widehat{L}) - \underline{L}_{ik} \\ & = & b^{\mathrm{pnp}}_{ik}(F) - \underline{L}_{ik}. \end{array}$$

Csóka and Herings (2021) showed that b^{pnp} satisfies invariance to mitosis, claims boundedness, limited liability, priority of creditors, and continuity. For the sake of completeness, these proofs are replicated in Appendix A.

($\Leftarrow$)

Let $F = (N, z, L) \in \mathcal{F}$ be a financial network and let b be a bankruptcy rule satisfying net impartiality, invariance to mitosis, claims boundedness, limited liability, priority of creditors, and continuity. We show that $b(F) - \underline{L}$ is a solution to the system of equations and inequalities (4.3). Since (4.3) has $b^{\text{pnp}}(F) - \underline{L}$ as its unique solution by Proposition 4.4, it follows that $b = b^{\text{pnp}}$.

Since $b(F) \geq 0$, it follows that $b(F) - \underline{L} \geq -\underline{L}$, so the inequalities in (4.3) are all satisfied, and we can restrict attention to the equalities in (4.3).

We first assume all the liabilities to be rational numbers.

We consider two main cases.

Case 1: $i, j \in N$, $\widehat{L}_{ij} = 0$.
If $j = i$, then we have that $L_{ii} = 0 = b_{ii}(F)$, where the second equality follows by claims boundedness.
If $j \neq i$, then $\widehat{L}_{ij} = \widehat{L}_{ii} = 0$, so by net impartiality we have that $b_{ij}(F) - \underline{L}_{ij} = b_{ii}(F) - \underline{L}_{ii} = 0$, where the last equality follows from the case $j = i$.

Case 2: $i, j \in N$, $\widehat{L}_{ij} > 0$.

Case 2a: $z_i + \sum_{k \in N} b_{ki}(F) \geq \sum_{k \in N} L_{ik}$.
It follows that

$$\begin{array}{rcl} z_i + \sum_{k \in N}(b_{ki}(F) - \underline{L}_{ki}) & = & z_i + \sum_{k \in N} b_{ki}(F) - \sum_{k \in N} \underline{L}_{ki} \\ & \geq & \sum_{k \in N} L_{ik} - \sum_{k \in N} \underline{L}_{ki} \\ & = & \sum_{k \in N} L_{ik} - \sum_{k \in N} \underline{L}_{ik} \\ & = & \sum_{k \in N} \widehat{L}_{ik}, \end{array}$$

so $\widehat{\Pi}_{ij}(z_i + \sum_{k \in N}(b_{ki}(F) - \underline{L}_{ki})) \geq \widehat{L}_{ij}$.

We therefore have to show that $b_{ij}(F) - \underline{L}_{ij} = \widehat{L}_{ij}$. Suppose, on the contrary, that $b_{ij}(F) - \underline{L}_{ij} \neq \widehat{L}_{ij}$. Then, by claims boundedness, we have that

$$b_{ij}(F) - \underline{L}_{ij} < \widehat{L}_{ij}. \tag{5.21}$$

By priority of creditors, the assumption of Case 2a, claims boundedness, and (5.21) we get that

$$0 = e_i(N, z, b(F)) = a_i(N, z, b(F)) - \textstyle\sum_{k \in N} b_{ik}(F) > \sum_{k \in N} L_{ik} - \sum_{k \in N} L_{ik} = 0, \tag{5.22}$$

a contradiction. Consequently, it holds that $b_{ij}(F) - \underline{L}_{ij} = \widehat{L}_{ij}$.

Case 2b: $z_i + \sum_{k \in N} b_{ki}(F) < \sum_{k \in N} L_{ik}$.
It follows that

$$\begin{array}{rcl} z_i + \sum_{k \in N}(b_{ki}(F) - \underline{L}_{ki}) & = & z_i + \sum_{k \in N} b_{ki}(F) - \sum_{k \in N} \underline{L}_{ki} \\ & < & \sum_{k \in N} L_{ik} - \sum_{k \in N} \underline{L}_{ki} \\ & = & \sum_{k \in N} L_{ik} - \sum_{k \in N} \underline{L}_{ik} \\ & = & \sum_{k \in N} \widehat{L}_{ik}, \end{array}$$

so $\widehat{\Pi}_{ij}(z_i + \sum_{k \in N}(b_{ki}(F) - \underline{L}_{ki})) < \widehat{L}_{ij}$.

We therefore have to show that $b_{ij}(F) - \underline{L}_{ij} = \widehat{\Pi}_{ij}(z_i + \sum_{k \in N}(b_{ki}(F) - \underline{L}_{ki}))$.

Since the liabilities are rational numbers, the net liabilities are all rational multiples of one another. We argue that we can use Lemma 5.1 to conclude that there exists a real number $r_i \geq 0$ such that

$$b_{ik}(F) - \underline{L}_{ik} = r_i \widehat{L}_{ik}, \quad k \in N. \tag{5.23}$$

By the assumption of Case 2, there is a $k' \in N$ such that $\widehat{L}_{ik'} > 0$. Let us define

$$r_i = \frac{b_{ik'}(F) - \underline{L}_{ik'}}{\widehat{L}_{ik'}}. \tag{5.24}$$

If $k \in N$ is such that $\widehat{L}_{ik} = 0$, then (5.23) follows from Case 1. Thus let $k \in N$ be such that $\widehat{L}_{ik} > 0$. By Lemma 5.1, we get that

$$b_{ik}(F) - \underline{L}_{ik} = \frac{\widehat{L}_{ik}}{\widehat{L}_{ik'}}(b_{ik'}(F) - \underline{L}_{ik'}) = r_i \widehat{L}_{ik},$$

so equation (5.23) is satisfied.

For $b(F) - \underline{L}$ to be a solution to the system of equations (4.3), it remains to be shown that

$$r_i = \frac{z_i + \sum_{k \in N}(b_{ki}(F) - \underline{L}_{ki})}{\sum_{k \in N} \widehat{L}_{ik}}. \tag{5.25}$$

By the assumption of Case 2b and limited liability, there is an agent $k \in N$ such that $b_{ik}(F) < L_{ik}$, so priority of creditors gives

$$\sum_{k \in N}(b_{ik}(F) - \underline{L}_{ik}) = z_i + \sum_{k \in N}(b_{ki}(F) - \underline{L}_{ki}). \tag{5.26}$$

By (5.23) and (5.26), we have that

$$\sum_{k \in N} r_i \widehat{L}_{ik} = z_i + \sum_{k \in N}(b_{ki}(F) - \underline{L}_{ki}), \tag{5.27}$$

implying (5.25).

We have shown that $b = b^{\mathrm{pnp}}$ when all liabilities in F are rational numbers.

We now consider the case where the liabilities in F are real-valued. Let $(F^n)_{n\in\mathbb{N}}$ be a sequence of financial networks where all liabilities are rational numbers, converging to F. We have that

$$b(F) = \lim_{n\to\infty} b(F^n) = \lim_{n\to\infty} b^{\mathrm{pnp}}(F^n) = b^{\mathrm{pnp}}(F),$$

where the first equality follows from the axiom of continuity, the second follows since the liabilities in F^n are all rational numbers, and the third since b^{pnp} satisfies continuity. $\square$

6 Independence of the Axioms

In this section, we show the independence of the axioms of net impartiality, invariance to mitosis, claims boundedness, limited liability, priority of creditors, and continuity by providing six examples of bankruptcy rules satisfying all the axioms except one.

We have seen that net impartiality is not satisfied by the proportional rule b^{p} in Example 2.3. It has been shown in Csóka and Herings (2021) that b^{p} satisfies the other axioms.

To present a bankruptcy rule that satisfies all axioms except invariance to mitosis, it is convenient to consider division rules for claims problems first. A *claims problem* $C = (N, E, c)$ consists of a set of agents $N \in \mathcal{N}$, a strictly positive *estate* $E > 0$, and a non-negative claims vector $c \in \mathbb{R}^N_+$. The set of all claims problems is denoted by $\mathcal{C}$. We define the set of vectors $\mathcal{V} = \cup_{N\in\mathcal{N}}\mathbb{R}^N_+$. A *division rule* is a function $d : \mathcal{C} \to \mathcal{V}$ such that for every $(N, E, c) \in \mathcal{C}$ it holds that $d(N, E, c) \in \mathbb{R}^N_+$ and $\sum_{i\in N} d_i(N, E, c) = \min\{E, \sum_{i\in N} c_i\}$. Moreover, a division rule is required to be monotonic. If $(N, E, c), (N, E', c) \in \mathcal{C}$ are such that $E \leq E'$, then $d(N, E, c) \leq d(N, E', c)$. A division rule d specifies how a given estate E should be allocated among a set of claimants and allocations are required to be increasing in the estate.

We can base a bankruptcy rule on division rules by treating the asset value of an agent as the estate and the vector of liabilities as the claims vector. More precisely, a division rule $d : \mathcal{C} \to \mathcal{V}$ induces the bankruptcy rule $b : \mathcal{F} \to \mathcal{M}$ by defining

$$b(N, z, L) = \max\{P \in \mathcal{M}(N) \mid \text{for every } i, j \in N,\ P_{ij} = d_j(a_i(P), L_i)\}, \quad (N, z, L) \in \mathcal{F}. \tag{6.1}$$

The proportional division rule d^{p} is obtained by defining, for every $(N, E, c) \in \mathcal{C}$, for every

$i \in N$,

$$d_i^{\mathrm{p}}(N,E,c) = \begin{cases} 0, & \text{if } c_i = 0, \\ \min\{\frac{c_i}{\sum_{j\in N} c_j}E, c_i\}, & \text{otherwise.} \end{cases}$$

The system of equations in (6.1), for every $i, j \in N$, $P_{ij} = d_j^{\mathrm{p}}(a_i(P), L_i)$ can be shown to have a unique solution which is equal to the solution of (2.1). The proportional division rule therefore induces the bankruptcy rule b^{p}. For other division rules d, the system of equations in (6.1), for every $i, j \in N$, $P_{ij} = d_j(a_i(P), L_i)$ can have multiple solutions, but the set of solutions can be shown to have a lattice structure, and the maximum operator in (6.1) therefore singles out a unique payment matrix. For details, we refer to Csóka and Herings (2023).

The constrained equal losses division rule imposes that all claimants face the same loss up to the value of their claim. For $(N, E, c) \in \mathcal{C}$, if $E > \sum_{i\in N} c_i$, then define $\mu = 0$. Otherwise, define $\mu \in [0, \max_{i\in N} c_i]$ as the unique solution to

$$\sum_{i\in N} \max\{c_i - \mu, 0\} = E.$$

The constrained equal losses division rule $d^{\mathrm{cel}} : \mathcal{C} \to \mathcal{V}$ is obtained by defining, for every $(N, E, c) \in \mathcal{C}$, for every $i \in N$,

$$d_i^{\mathrm{cel}}(N, E, c) = c_i - \mu.$$

The system of equations in (6.1), for every $i, j \in N$, $P_{ij} = d_j^{\mathrm{cel}}(a_i(P), L_i)$ can be shown to have a unique solution, which generates the constrained equal losses bankruptcy rule b^{cel}.

The pairwise netting constrained equal losses bankruptcy rule is defined as follows.

Definition 6.1. The *pairwise netting constrained equal losses rule* $b^{\mathrm{pncel}} : \mathcal{F} \to \mathcal{M}$ is defined by

$$b^{\mathrm{pncel}}(N, z, L) = \underline{L} + b^{\mathrm{cel}}(N, z, \widehat{L}), \quad (N, z, L) \in \mathcal{F}.$$

The following example shows that the pairwise netting constrained equal losses rule violates invariance to mitosis.

Example 6.2 (The pairwise netting constrained equal losses rule)**.** Consider a financial network $(N, z, L) \in \mathcal{F}$ with three agents $N = \{1, 2, 3\}$. Table 3 presents the initial endowments z, the liabilities L, the net liabilities $\widehat{L}$, the payment matrix $\widehat{P} = b^{\mathrm{cel}}(N, z, \widehat{L})$ resulting from the constrained equal losses rule using the net liabilities, the payment matrix $P = b^{\mathrm{pncel}}(N, z, L)$ resulting from the pairwise netting constrained equal losses rule, asset values $a(N, z, P)$, and equities $e(N, z, P)$. Note that agent 1 applies a haircut of 3 units to all its liabilities, i.e., $\mu_1 = 3$.

z	L			$\widehat{L}$			$\widehat{P}$			P			$a(N,z,P)$	$e(N,z,P)$
18	0	8	18	0	8	16	0	5	13	0	5	15	20	0
2	0	0	0	0	0	0	0	0	0	0	0	0	7	7
2	2	0	0	0	0	0	0	0	0	2	0	0	17	15

Table 3: The initial endowments z, the liabilities L, the net liabilities $\widehat{L}$, the payment matrix $\widehat{P} = b^{\mathrm{cel}}(N, z, \widehat{L})$, the payment matrix $P = b^{\mathrm{pncel}}(N, z, L)$, asset values $a(N, z, P)$, and equities $e(N, z, P)$.

Now consider the financial network $F' = (N', z', L') = (N \cup \{4\}, z', L')$ obtained by splitting agent 3 into identical agents 3 and 4. Table 4 presents the resulting initial endowments z', the liabilities L', the net liabilities $\widehat{L}'$, the payment matrix $\widehat{P}' = b^{\mathrm{cel}}(N', z', \widehat{L}')$ resulting from the constrained equal losses rule using the net liabilities, the payment matrix $P' = b^{\mathrm{pncel}}(N', z', L')$ resulting from the pairwise netting constrained equal losses rule, asset values $a(N', z', P')$, and equities $e(N', z', P')$.

z'	L'				$\widehat{L}'$				$\widehat{P}'$				P'				$a(N',z',P')$	$e(N',z',P')$
18	0	8	9	9	0	8	8	8	0	6	6	6	0	6	7	7	20	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8	8
1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	8	7
1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	8	7

Table 4: The initial endowments z', the liabilities L', the net liabilities $\widehat{L}'$, the payment matrix $\widehat{P}' = b^{\mathrm{cel}}(N', z', \widehat{L}')$, the payment matrix $P' = b^{\mathrm{pncel}}(N', z', L')$, asset values $a(N', z', P')$, and equities $e(N', z', P')$.

Then $P_{13} = 15 > 14 = P'_{13} + P'_{14}$, so b^{pncel} does not satisfy invariance to mitosis. △

It has been shown in Csóka and Herings (2023) that b^{cel} is continuous. The continuity of b^{pncel} then follows. It is easy to verify that b^{cel} satisfies the axioms of impartiality, claims boundedness, limited liability, and priority of creditors. It then follows that b^{pncel} satisfies the axioms of net impartiality, claims boundedness, limited liability, and priority of creditors.

Definition 6.3. The *pairwise netting proportional with double liabilities rule* $b^{\mathrm{pnpd}} : \mathcal{F} \to \mathcal{M}$ is defined by

$$b^{\mathrm{pnpd}}(N, z, L) = \underline{L} + b^{\mathrm{p}}(N, z, 2\widehat{L}), \quad (N, z, L) \in \mathcal{F}.$$

The pairwise netting proportional with double liabilities rule b^{pnpd} performs one round of pairwise netting followed by an application of the proportional rule, but pretends that the net liabilities are twice the actual net liabilities. It is obvious that b^{pnpd} does not satisfy the axiom of claims boundedness.

Let $F = (N, z, L) \in \mathcal{F}$ and $i, j, k \in N$ be such that $L_{ij} - \underline{L}_{ij} = L_{ik} - \underline{L}_{ik}$. Since $2\widehat{L}_{ij} = 2L_{ij} - 2\underline{L}_{ij} = 2L_{ik} - 2\underline{L}_{ik} = 2\widehat{L}_{ik}$ and b^{p} satisfies impartiality, we have that

$$\begin{aligned} b^{\mathrm{pnpd}}_{ij}(N, z, L) - \underline{L}_{ij} &= \underline{L}_{ij} + b^{\mathrm{p}}_{ij}(N, z, 2\widehat{L}) - \underline{L}_{ij} \\ &= \underline{L}_{ij} + b^{\mathrm{p}}_{ik}(N, z, 2\widehat{L}) - \underline{L}_{ij} \\ &= \underline{L}_{ik} + b^{\mathrm{p}}_{ik}(N, z, 2\widehat{L}) - \underline{L}_{ik} \\ &= b^{\mathrm{pnpd}}_{ik}(N, z, L) - \underline{L}_{ik}, \end{aligned}$$

so b^{pnpd} satisfies net impartiality.

Let $F = (N, z, L) \in \mathcal{F}$ and $i \in N$ be such that $b^{\mathrm{pnpd}}_{i}(F) < L_i$, so $\underline{L}_i + b^{\mathrm{p}}_{i}(N, z, 2\widehat{L}) < L_i$ or, equivalently, $b^{\mathrm{p}}_{i}(N, z, 2\widehat{L}) < \widehat{L}_i$. This implies $b^{\mathrm{p}}_{i}(N, z, 2\widehat{L}) < 2\widehat{L}_i$. Since b^{p} satisfies priority of creditors, we have that $e_i(N, z, b^{\mathrm{p}}(N, z, 2\widehat{L})) = 0$. It follows that

$$\begin{aligned} e_i(N, z, b^{\mathrm{pnpd}}(F)) &= e_i(N, z, \underline{L} + b^{\mathrm{p}}(N, z, 2\widehat{L})) \\ &= z_i + \textstyle\sum_{j \in N}(\underline{L}_{ji} + b^{\mathrm{p}}_{ji}(N, z, 2\widehat{L}) - \underline{L}_{ij} - b^{\mathrm{p}}_{ij}(N, z, 2\widehat{L})) \\ &= z_i + \textstyle\sum_{j \in N}(b^{\mathrm{p}}_{ji}(N, z, 2\widehat{L}) - b^{\mathrm{p}}_{ij}(N, z, 2\widehat{L})) \\ &= e_i(N, z, b^{\mathrm{p}}(N, z, 2\widehat{L})) = 0, \end{aligned}$$

so b^{pnpd} satisfies priority of creditors.

We have already argued that the proportional rule satisfies the axioms of invariance to mitosis, limited liability, and continuity. It then follows easily that b^{pnpd} satisfies these axioms as well.

Definition 6.4. The *all liabilities are paid rule* $b^{\mathrm{all}} : \mathcal{F} \to \mathcal{M}$ is defined by

$$b^{\mathrm{all}}(N, z, L) = L, \quad (N, z, L) \in \mathcal{F}.$$

The payment matrix resulting from b^{all} equals the liability matrix. It is obvious that b^{all} does not satisfy the axiom of limited liability. It is easy to show that b^{all} satisfies the axioms of net impartiality, invariance to mitosis, claims boundedness, priority of creditors, and continuity.

Definition 6.5. The *common liabilities are paid rule* $b^{\mathrm{cl}} : \mathcal{F} \to \mathcal{M}$ is defined by

$$b^{\mathrm{cl}}(N, z, L) = \underline{L}, \quad (N, z, L) \in \mathcal{F}.$$

The payment matrix of the common liabilities are paid rule b^{cl} equals the matrix $\underline{L}$. It is obvious that b^{cl} does not satisfy priority of creditors.

Let $F = (N, z, L) \in \mathcal{F}$ and $i, j, k \in N$ be such that $L_{ij} - \underline{L}_{ij} = L_{ik} - \underline{L}_{ik}$. We have that

$$b^{\mathrm{cl}}_{ij}(N, z, L) - \underline{L}_{ij} = \underline{L}_{ij} - \underline{L}_{ij} = \underline{L}_{ik} - \underline{L}_{ik} = b^{\mathrm{cl}}_{ik}(N, z, L) - \underline{L}_{ik},$$

so b^{cl} satisfies net impartiality.

It is easy to show that b^{cl} satisfies the axioms of invariance to mitosis, claims boundedness, limited liability, and continuity.

We finally define the proportional with priority to irrational claims division rule $d^{\mathrm{pi}} : \mathcal{C} \to \mathcal{V}$. Let $(N, E, c) \in \mathcal{C}$. We denote the players with rational claims by $N^{\mathrm{ra}} = \{i \in N \mid c_i \in \mathbb{Q}\}$ and those with irrational claims by $N^{\mathrm{ir}} = \{i \in N \mid c_i \in \mathbb{R} \setminus \mathbb{Q}\}$. We divide the players in two priority classes, N^{ra} and N^{ir}, where players in N^{ir} have priority over those in N^{ra}, and apply the proportional rule within each priority class. More precisely, we have that

$$d^{\mathrm{pi}}(N, E, c) = \begin{cases} 0, & \text{if } c_i = 0, \\ \min\{\frac{c_i}{\sum_{j \in N^{\mathrm{ir}}} c_j} E, c_i\}, & \text{if } i \in N^{\mathrm{ir}} \text{ and } c_i > 0, \\ \min\{\frac{c_i}{\sum_{k \in N^{\mathrm{ra}}} c_k} \max\{E - \sum_{j \in N^{\mathrm{ir}}} c_j, 0\}, c_i\}, & \text{if } i \in N^{\mathrm{ra}} \text{ and } c_i > 0. \end{cases}$$

The system of equations in (6.1), for every $i, j \in N$, $P_{ij} = d_j^{\mathrm{pi}}(a_i(P), L_i)$ has a set of solutions with a lattice structure. Taking the maximal solution leads to the proportional with priority to irrational claims bankruptcy rule b^{pi}.

Definition 6.6. The *pairwise netting proportional with priority to irrational claims rule* $b^{\mathrm{pnpi}} : \mathcal{F} \to \mathcal{M}$ is defined by

$$b^{\mathrm{pnpi}}(N, z, L) = \underline{L} + b^{\mathrm{pi}}(N, z, \widehat{L}), \quad (N, z, L) \in \mathcal{F}.$$

The pairwise netting proportional with priority to irrational claims rule does not satisfy continuity. For instance, let $(N, z, L) \in \mathcal{F}$ be such that $N = \{1, 2, 3\}$, $0 < L_{12} \in \mathbb{Q}$, and $0 < L_{13} \in \mathbb{R} \setminus \mathbb{Q}$. For $n \in \mathbb{N}$, let $(F^n)_{n \in \mathbb{N}} = (N^n, z^n, L^n)_{n \in \mathbb{N}}$ be a sequence of financial networks of $\mathcal{F}$ such that, for every $n \in N$, $N^n = N$, $z^n = z$, $L_2^n = L_2$, $L_3^n = L_3$, L_1^n has only entries in $\mathbb{Q}$, and $L_1^n \to L_1$. At the limit L, agent 1 gives priority to creditor 3, but at every L^n both creditors 2 and 3 are given equal priority. It now follows easily that b^{pnpi} violates continuity. As a concrete numerical example, take $z_1 = z_2 = z_3 = 1$, $L_{12} = 1$, $L_{13} = \sqrt{2}$, $L_2 = 0$, and $L_3 = 0$. Let $P^n = b^{\mathrm{pnpi}}(N, z, L^n)$ and $P = b^{\mathrm{pnpi}}(N, z, L)$. It holds that $\lim_{n \to \infty} P_{12}^n = 1/(1 + \sqrt{2})$, whereas $P_{12} = 0$.

Since b^{pi} satisfies impartiality, it follows that b^{pnpi} satisfies net impartiality. Since dividing a rational number by a natural number results in a rational number and dividing an irrational number by a natural number leads to an irrational number, the splitting of an agent in multiple identical agents results in multiple identical agents in the same priority class as the original agent. Invariance to mitosis of b^{pnpi} now follows from similar arguments as invariance to mitosis of b^{pnp}. Also the proof that b^{pnpi} satisfies the axioms of claims boundedness, limited liability, and priority to creditors follows along the same lines as the proof that b^{pnp} satisfies those axioms.

Table 5 summarizes the bankruptcy rules and the axioms that they satisfy. The symbol $\surd$ means that the bankruptcy rule satisfies the given axiom of the column and $\neg$ that it does not.

	N	I	B	L	P	C
Pairwise netting proportional	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
Proportional	$\neg$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
Pairwise netting constrained equal losses	$\surd$	$\neg$	$\surd$	$\surd$	$\surd$	$\surd$
Pairwise netting proportional with double liabilities	$\surd$	$\surd$	$\neg$	$\surd$	$\surd$	$\surd$
All liabilities are paid	$\surd$	$\surd$	$\surd$	$\neg$	$\surd$	$\surd$
Common liabilities are paid	$\surd$	$\surd$	$\surd$	$\surd$	$\neg$	$\surd$
Pairwise netting proportional with priority to irrational claims	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\neg$

Table 5: Bankruptcy rules and the axioms of net impartiality (N), invariance to mitosis (I), claims boundedness (B), limited liability (L), priority of creditors (P), and continuity (C).

7 Conclusion

Financial networks consider situations where agents are linked to each other via mutual liabilities. A commonly applied rule to clear financial networks in case of bankruptcy is the proportional rule. This rule has rather unappealing features. In particular, it may happen that an agent with a strictly positive endowment and strictly positive net claims on any other agent, still becomes insolvent when the proportional rule is applied. Such an undesirable feature is avoided when clearing starts out with one round of pairwise netting of mutual liabilities and the proportional rule is applied next to the net liabilities. The resulting bankruptcy rule is called the pairwise netting proportional rule.

The pairwise netting proportional rule satisfies a number of attractive axioms. Net impartiality requires that two creditors with the same net claims on a third agent, receive the same payment from that agent. Invariance to mitosis imposes that splitting an agent into a number of identical agents, where splitting affects initial endowments, claims, and liabilities, does not affect the payments between any two agents not involved in the split and does not affect the sum of payments received from and made to the agents involved in the split by any other agent. Three more axioms reflect standard accounting principles. Claims boundedness expresses that no agent receives more than its claim on another agent. Limited liability states that agents end up with non-negative equity. Priority to creditors means that default is only allowed when no equity is left. Finally, there is the axiom of continuity, which states the desirable property that small perturbations to a financial network result to small changes in payments.

This paper shows that the pairwise netting proportional rule satisfies net impartiality, invariance to mitosis, claims boundedness, limited liability, priority to creditors, and continuity and is in fact the only bankruptcy rule to do so.

A Proof that b^{pnp} satisfies invariance to mitosis, claims boundedness, limited liability, priority of creditors, and continuity.

To show that b^{pnp} satisfies invariance to mitosis, we define the bankruptcy rules $\underline{b} : \mathcal{F} \to \mathcal{M}$ and $b^{\widehat{\mathrm{p}}} : \mathcal{F} \to \mathcal{M}$ by setting, for $F = (N, z, L) \in \mathcal{F}$, $\underline{b}(F) = \underline{L}$ and $b^{\widehat{\mathrm{p}}}(F) = b^{\mathrm{p}}(N, z, \widehat{L})$. It holds that $b^{\mathrm{pnp}}(F) = \underline{b}(F) + b^{\widehat{\mathrm{p}}}(F)$. We show that both $\underline{b}$ and $b^{\widehat{\mathrm{p}}}$ satisfy invariance to mitosis, from which it follows that b^{pnp} satisfies invariance to mitosis.

Let $F = (N, z, L) \in \mathcal{F}$ and $F' = (N', z', L') \in \mathcal{F}$ be financial networks as defined in Axiom 3.3, so endowments and liabilities of agent $j \in N$ are divided equally over the agents in $\{j\} \cup K$. For every $h, i \in N \setminus \{j\}$ it holds that

$$\begin{aligned}
\textstyle\sum_{k \in \{j\} \cup K} \underline{b}_{ki}(F') &= \textstyle\sum_{k \in \{j\} \cup K} \underline{L}'_{ki} = \sum_{k \in \{j\} \cup K} \frac{1}{|K|+1} \underline{L}_{ji} = \underline{L}_{ji} = \underline{b}_{ji}(F), \\
\textstyle\sum_{k \in \{j\} \cup K} \underline{b}_{ik}(F') &= \textstyle\sum_{k \in \{j\} \cup K} \underline{L}'_{ik} = \sum_{k \in \{j\} \cup K} \frac{1}{|K|+1} \underline{L}_{ij} = \underline{L}_{ij} = \underline{b}_{ij}(F), \\
b_{hi}(F') &= \underline{L}'_{hi} = \underline{L}_{hi} = b_{hi}(F),
\end{aligned}$$

so $\underline{b}$ satisfies invariance to mitosis.

As also the net liabilities $\widehat{L}$ are divided equally over the agents in $\{j\} \cup K$ and b^{p} satisfies invariance to mitosis, it follows that $b^{\widehat{\mathrm{p}}}$ satisfies invariance to mitosis.

Claims boundedness and continuity is obviously satisfied by b^{pnp}.

To check that b^{pnp} satisfies limited liability, consider a financial network $F = (N, z, L) \in \mathcal{F}$ and any agent $i \in N$. Then

$$\begin{aligned}
e_i(N, z, b^{\mathrm{pnp}}(F)) &= z_i + \textstyle\sum_{j \in N} b^{\mathrm{pnp}}_{ji}(F) - \sum_{j \in N} b^{\mathrm{pnp}}_{ij}(F) \\
&= z_i + \textstyle\sum_{j \in N} \underline{L}_{ji} + \sum_{j \in N} b^{\mathrm{p}}_{ji}(N, z, \widehat{L}) \\
&\quad - \textstyle\sum_{j \in N} \underline{L}_{ij} - \sum_{j \in N} b^{\mathrm{p}}_{ij}(N, z, \widehat{L}) \\
&= e_i(N, z, b^{\mathrm{p}}(N, z, \widehat{L})) \geq 0,
\end{aligned} \tag{A.1}$$

since $\underline{L}_{ij} = \underline{L}_{ji}$ for all $i, j \in N$, and the proportional rule b^{p} satisfies limited liability. Thus b^{pnp} satisfies limited liability.

To verify that b^{pnp} satisfies priority of creditors, consider a financial network $F = (N, z, L) \in \mathcal{F}$ and any agent $i \in N$ such that $b^{\mathrm{pnp}}_i(F) < L_i$, implying that

$$b^{\mathrm{p}}_i(N, z, \widehat{L}) < \widehat{L}_i. \tag{A.2}$$

Since b^{p} satisfies priority of creditors, (A.2) implies that

$$e_i(N, z, b^{\text{p}}(N, z, \widehat{L})) = 0.$$

Using the same argument as in (A.1), it follows that $e_i(N, z, b^{\text{pnp}}(F)) = 0$, thus b^{pnp} satisfies priority of creditors. □

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