Why (and When) Are Preferences Convex? Threshold Effects and Uncertain Quality*

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Abstract

It is often assumed (for analytical convenience, but also in accordance with common intuition) that consumer preferences are convex. In this paper, we consider circumstances under which such preferences are (or are not) optimal. In particular, we investigate a setting in which goods possess some hidden quality with known distribution, and the consumer chooses a bundle of goods that maximizes the probability that he receives some threshold level of this quality. We show that if the threshold is small relative to consumption levels, preferences will tend to be convex; whereas the opposite holds if the threshold is large. Our theory helps explain a broad spectrum of economic behavior (including, in particular, certain common commercial advertising strategies), suggesting that sensitivity to information about thresholds is deeply rooted in human psychology.

1 Introduction

Convexity of preferences is one of a small handful of canonical assumptions in economic theory. Typically justified in introductory texts by a brief appeal to introspection, convexity is appealing in part because it is conducive to marginal analysis and to single-valued, continuous demand functions.[1] But in the

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[1] Such brevity is not limited to introductory textbooks. The popular graduate text of Mas-Colell, Whinston, and Green (1995, p. 44) justifies the convexity assumption as follows:

real world, there are many situations in which sudden shifts in demand are observed[2], and the consumer's "preference for variety" finds its limits. It would therefore be useful to have a theory of preferences in which convexity (and its counterpart, nonconvexity) arises as the predictable result of a well-defined choice environment.

Given the ubiquity of convex preferences in economic models, it is surprising how little scrutiny this particular aspect of human nature has received.[3] While there have been many refinements to the theory of convex preferences (e.g., Kannai 1977, Richter and Wong 2004), these authors always ultimately *assume* the primitive behavioral postulate in question, rather than asking the deeper question of the circumstances in which such preferences might be *optimal*.[4] Our approach to this question will be to explicitly step back from reliance on the consumer's subjective report of his motivations for choosing to purchase particular goods (a notoriously unreliable method of inquiry, if modern neuroscience is to be believed),[5] and appeal instead to evidence from fields such as social psychology, marketing, and behavioral ecology. The first innovation to flow from this dismissal of subjective experience will quickly become apparent: in developing the beginnings of a normative theory of convex preferences, we will treat as *stochastic* decision problems that have typically been viewed as deterministic. It is certainly true that a college student deciding what combination of bread and soup to consume at lunchtime is unlikely to view his decision as involving a risky portfolio of uncertain inputs (nutrients, pathogens, etc.) and unknown outcomes (health, survival), but viewed as a problem with an objective optimum–i.e., one in which the combined wisdom of human evolutionary history sees a multitude of possible outcomes, each with a well-defined payoff–that is exactly what it is.[6]

"A taste for diversification is a realistic trait of economic life. Economic theory would be in serious difficulty if this postulated propensity for diversification did not have significant descriptive content."

[2]An important example of such shifts in individual demand is the consumer response to product advertisements. This phenomenon is typically viewed in economics as driven by informative signaling (see, e.g., Milgrom and Roberts (1986)) rather than nonconvexities, but the two explanations are not necessarily inconsistent. Indeed, the framework we will develop emphasizes the role of information in inducing non-convex behavior, and–as we note below–even provides a rich framework for predicting ad content.

[3]It should be noted that this lack of scrutiny does not pertain to the producer-theoretic analog of our question: the monopolist's optimal bundling decision. See, for example, Fang and Norman (2006) or Ibragimov (2005).

[4]A partial exception is found in the theory of risk-bearing, in which it has long been known that risk-averse consumers (i.e., consumers with concave expected utility functions–implying convex preferences over wealth outcomes) should choose a diversified portfolio (Arrow 1971). This result is closely related to the theory we will develop below, but again it requires assumption of the basic postulate (risk aversion).

[5]See, for instance, Gazzaniga (2000). For a broad review of findings in social psychology and neuroscience that relate to the development of economic and legal theory, see Hanson and Yosifon (2004).

[6]Interestingly, while we choose a stochastic framework for decision problems that are commonly perceived to be deterministic, Huffaker (1998) has argued the opposite: that most stochastic environments are driven by deterministic underlying physical phenomena, and thus can (and perhaps should) be modeled deterministically. We hope the reader will not see a

It is not difficult to imagine situations in the evolutionary history of our species in which a propensity for diversity in consumption would have been beneficial. An obvious example is nutrition: the human diet must include a host of essential nutrients (from calories and protein to iron and Vitamin C) in order to sustain life, but no particular food contains all these nutrients in the necessary proportions. Similarly, in societies in which food sharing is an important form of social insurance, it might pay to distribute favors among many allies, rather than directing them to just a few close friends (Kaplan and Gurven 2005). Or if loss of social status–an important form of "wealth" in many pre-industrial societies–is a concern, then it might make sense to avoid risky gambles involving large, conspicuous losses. The common theme we see in these examples is the presence of threshold payoffs: in the natural world, the consequences of consuming insufficient quantities of limiting micronutrients include debilitating illness and death; going an extended period of time without food results in starvation; and social status is by nature a relative measure, and–in the small groups that characterized most of human history–necessarily entails discrete thresholds with respect to rank order. We will argue in this paper that threshold payoffs provide a foundation upon which to build a theory of convex preferences that captures certain deeply rooted aspects of consumer psychology. Moreover (we argue) the implications of our theory are consistent with the content and form of many modern marketing messages delivered by profit-maximizing firms. This last observation underscores an important advantage of the theory of threshold utility we introduce, should it prove to have some generality: for the same reasons that human evolution presumably favored strong reactions to the presence of thresholds (i.e., their stark consequences and ease of detection) make them promising subjects for empirical study in economics.

2 Utility in the presence of a quality threshold

Our starting point for this investigation is the problem studied by Smith and Tasnadi (2007) in the context of dietary habit formation. The setting is as follows: an individual chooses a bundle of two foods with uncertain concentrations of some limiting micronutrient. Given a finite gut size and known nutrient distributions, the objective was to choose the combination of foods that maximize the probability that nutrient intake meets a critical threshold level–which, given the context, was assumed to be "small" relative to the amounts of food consumed. We extend this work below by studying the more general problem of optimal consumption in the presence of uncertain product quality, and by considering "large" as well as "small" thresholds in the objective function. Indeed, we will show that the size of the threshold relative to consumption is a critical determinant of the convexity (and nonconvexity) of preferences.

contradiction in our agreement with this view. Indeed, our stopping point in this paper–i.e., the probability distributions, which we will treat as exogenous–would seem a natural next step for future research. For a discussion of the merits of pursuing ever-deeper levels of causation in the social sciences, see Smith (2009).

We now formalize the problem as follows. A decision-maker ("consumer") is faced with a menu of two goods, x and y, and must choose how much of each to consume, given income m and prices p and 1, respectively. There is a single unobservable characteristic (quality) for which there is a critical threshold: the consumer seeks only to maximize the probability that he consumes k units of this quality. The amounts of the unobservable quality per unit of x and y are independent random variables, denoted C_x and C_y, with distribution functions F and G, respectively. Formally, the consumer's utility function is given by

$$U(x,y) = P(C_x x + C_y y \geq k), \tag{1}$$

and his decision problem can be stated as follows:

$$\begin{aligned} \max_{x,y} \quad & U(x,y) \\ \text{s.t.} \quad & px + y \leq m \\ & x, y \geq 0 \end{aligned} \tag{2}$$

If the support of these random variables is the unit interval,[7] then (assuming continuous random variables with respective density functions f and g)[8] determination of the consumer's utility function

$$U(x,y) = P(C_x x + C_y y \geq k) = \int_k^{\infty} \int_{\max\{0,t-y\}}^{\min\{x,t\}} \frac{1}{xy} f\left(\frac{z}{x}\right) g\left(\frac{t-z}{y}\right) dz dt$$

requires integration across five distinct regions in commodity space, which we illustrate in Figure 1.

We will refer to these regions as follows: the "death zone"

$$A^0 = \left\{(x,y) \in \mathbb{R}_+^2 \mid x + y \leq k\right\}$$

in which the probability of meeting the threshold is zero, the low-probability region

$$A^{--} = \left\{(x,y) \in \mathbb{R}_+^2 \mid k < x + y, x \leq k, y \leq k\right\}$$

in which probability of meeting the threshold is positive but the consumption levels of both goods are small (i.e., $x, y \leq k$), the region

$$A^{-+} = \left\{(x,y) \in \mathbb{R}_+^2 \mid x \leq k, k < y\right\}$$

in which the consumption level of x is small, the region

$$A^{+-} = \left\{(x,y) \in \mathbb{R}_+^2 \mid k < x, y \leq k\right\}$$

in which the consumption level of y is small, and the region

$$A^{++} = \left\{(x,y) \in \mathbb{R}_+^2 \mid k < x, k < y\right\}$$

in which the consumption levels of both x and y are large relative to the size of the threshold.

[7] If both random variables are non-negative and have finite support, this is just a matter of normalization.

[8] Note that independence of f and g implies that the density of the sum of the random variables $C_x x$ and $C_y y$ will be the convolution of their densities.

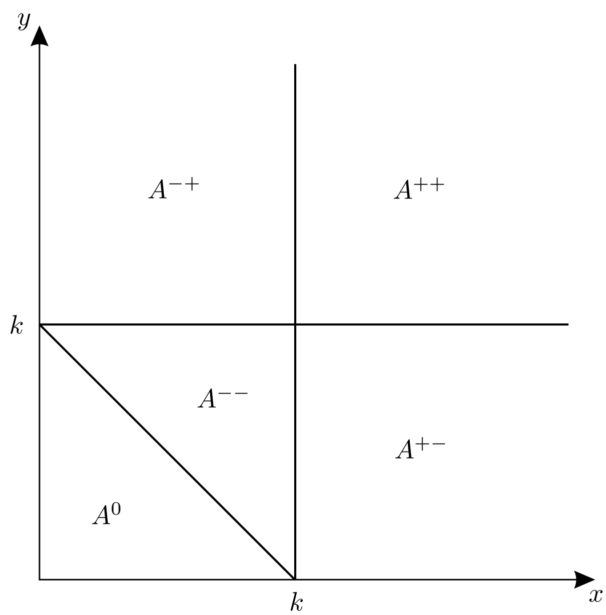

Figure 1: Five Regions

3 An Informative Special Case: Uniform Distributions

In order to isolate the effects of the threshold on the consumer's behavior, we begin by assuming that the random variables C_x and C_y are distributed according to the uniform distribution on the unit interval, i.e.,

Case 1 *"Uniform case":*

$$F(x) = G(x) = \begin{cases} 0, & \text{if } x < 0; \\ x, & \text{if } x \in [0,1]; \\ 1, & \text{if } 1 < x. \end{cases}$$

The following proposition summarizes the properties of utility function (1).

Proposition 1 *In Case 1, the consumer's utility function is quasi-concave on* $\{(x,y) \in \mathbb{R}_+^2 \mid 2k \le x + y\}$ *and quasi-convex on* $\{(x,y) \in \mathbb{R}_+^2 \mid k \le x + y \le 2k\}$. *Moreover, the indifference curves that describe his preferences on* $\mathbb{R}_+^2 \setminus A^0$ *are continuously differentiable, reflection invariant in the line* $y = x$, *strictly concave on* A^{--}, *strictly convex on* A^{++} *and linear on* A^{-+} *and* A^{+-}.

Proof. The utility function for Case 1 has been derived in a more general setting in Smith and Tasnadi (2007). Taking the symmetric setting into consideration we obtain for Case 1 that

$$U(x,y)=\begin{cases} 0 & \text{if} \quad (x,y)\in A^{0}, \\ 1-\frac{k}{x}+\frac{y}{2x}+\frac{(k-x)^2}{2xy} & \text{if} \quad (x,y)\in A^{--}, \\ 1+\frac{x}{2y}-\frac{k}{y} & \text{if} \quad (x,y)\in A^{-+}, \\ 1+\frac{y}{2x}-\frac{k}{x} & \text{if} \quad (x,y)\in A^{+-}, \\ 1-\frac{k^2}{2xy} & \text{if} \quad (x,y)\in A^{++}. \end{cases}$$

The indifference curves of utility function U are shown in Figure 2.

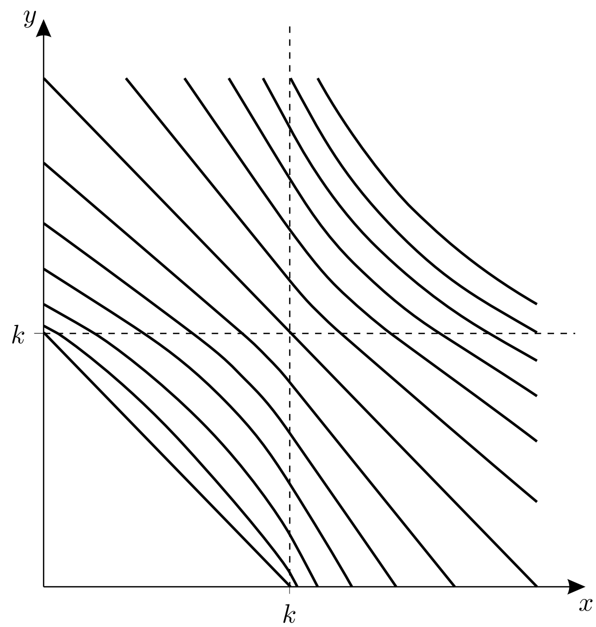

Figure 2: Indifference Curves ($k = 1$)

It can be easily verified that the indifference curves for U are strictly concave in A^{--}, strictly convex in A^{++} and linear in A^{-+} and A^{+-}. Moreover, the indifference curves associated with positive threshold probabilities are continuously differentiable and reflection invariant in the line $y = x$ by the symmetric setting. ■

Now we turn to problem (2) in Case 1.

Proposition 2 *The optimal solution of problem (2) in Case 1 is given by*

$(x^*, y^*) \in$

$$\begin{cases} \left\{\left(\frac{m}{2p}, \frac{m}{2}\right)\right\} & \text{if } \frac{m}{2p} > k \text{ and } p \geq 1; \\ \{(0, m)\} & \text{if } \frac{m}{2p} < k \text{ and } p > 1; \\ \left\{(x, y) \mid \lambda\left(\frac{m}{2p}, \frac{m}{2}\right) + (1-\lambda)(0, m), \lambda \in [0,1]\right\} & \text{if } \frac{m}{2p} = k \text{ and } p > 1; \\ \left\{\left(\frac{m}{2}, \frac{pm}{2}\right)\right\} & \text{if } \frac{pm}{2} > k \text{ and } p < 1; \\ \{(m, 0)\} & \text{if } \frac{pm}{2} < k \text{ and } p < 1; \\ \left\{(x, y) \mid \lambda\left(\frac{m}{2}, \frac{pm}{2}\right) + (1-\lambda)(m, 0), \lambda \in [0,1]\right\} & \text{if } \frac{pm}{2} = k \text{ and } p < 1; \\ \{(0, m), (m, 0)\} & \text{if } \frac{m}{2} < k \text{ and } p = 1; \\ \{(m-\lambda, \lambda) \mid \lambda \in [0,1]\} & \text{if } \frac{m}{2} = k \text{ and } p = 1. \end{cases}$$

Proof. We determine the optimal solution of Problem (2) as a function of k. Observe that the consumer's utility function is strongly monotonic in $\mathbb{R}^2_+ \setminus A^0$, and therefore, the optimal solution (x^*, y^*) lies on the budget line, i.e., $px^* + y^* = m$.

We start with the case $p \geq 1$. Hence, we can assume that $m > k$, since otherwise, the budget set is contained in A^0 and the consumer attains the threshold with probability zero. First, we assume that $(x^*, y^*) \in A^{++}$. Since the indifference curves in A^{++} are strictly convex we just have to find the indifference curve that has a unique intersection point with the budget line. Thus, we are searching for the utility level u for which

$$y = \frac{1}{2}\frac{k^2}{x(1-u)} = m - px$$

has a unique solution. From this we get

$$u^* = 1 - \frac{2pk^2}{m^2}, \quad x^* = \frac{m}{2p}, \quad \text{and} \quad y^* = \frac{m}{2}. \tag{3}$$

Because (x^*, y^*) has to be in A^{++} we must have $m/(2p) > k$ and $m/2 > k$ from which the first inequality is binding by $p \geq 1$. Second, it can be verified that if $m/(2p) < k$ and $p > 1$, then we have a corner solution in A^{-+}. In particular, $x^* = 0$ and $y^* = m$. Third, if $m/(2p) < k$ and $p = 1$, then the two corner solutions are $x^* = 0, y^* = m$ and $x^* = m, y^* = 0$. Forth, if $m/(2p) = k$ and $p > 1$, then the continuum of optimal solutions is given by the line segment connecting points $(0, m)$ and $(k, m/2)$. Fifth, if $m/(2p) = k$ and $p = 1$, then the continuum of optimal solutions is given by the line segment connecting points $(0, m)$ and $(0, m)$.

Finally, taking the symmetric setting into consideration one can obtain the solutions for the case of $p < 1$ simply by exchanging the roles played by variables x and y in our calculations. ■

We illustrate the optimal solutions as a function of k in Figure 3 if $p > 1$. Note that there is a discontinuous change in the demand correspondence as the threshold k increases from slightly lower values than $m/(2p)$ to slightly higher values than $m/(2p)$. For fixed k, this implies in turn that demand will be discontinuous in (p, m).[9]

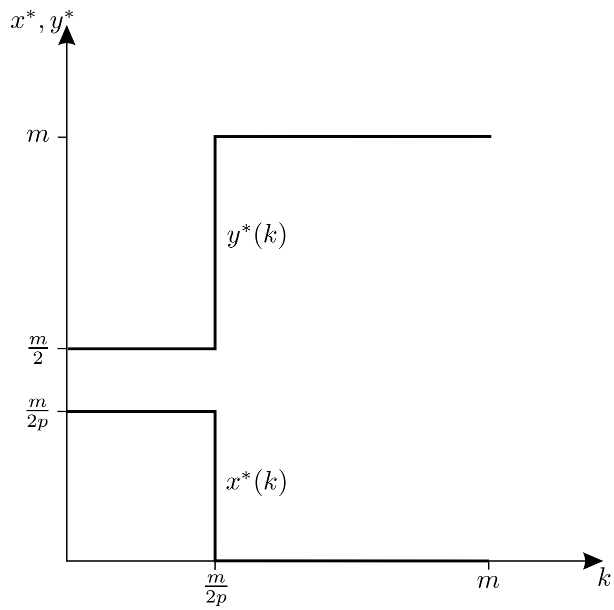

Figure 3: Optimal solutions for Case 1

As a corollary to Proposition 2 we are able to formulate our first result on discontinuous threshold effects.

Corollary 1 *In Case 1, we observe a discontinuous change in behavior at* $k = m/(2p)$ *if* $p \geq 1$ *and at* $k = pm/2$ *if* $p < 1$.

In this section, we have established that in the case of uniformly distributed quality in the presence of a threshold payoff, preferences will be strictly convex if the threshold is "small enough" relative to the amount of x and y consumed (i.e., for bundles in region A^{++}), and strictly non-convex if the threshold is "large enough" (i.e., for bundles in region A^{--}). And importantly, under these circumstances demand is a discontinuous function of price. Before turning to a more general class of distributions, in the next section we provide an intuitive explanation for these results.

[9] The demand correspondence in Case 1 is nevertheless *upper hemi-continuous* at $m = 2pk$, a property that does not require convexity of preferences (see, e.g., Mas-Colell, Whinston, and Green (1995), Propositions 3.D.2(iii) and 3.AA.1).

4 The Geometry of Threshold Effects

Under the current rules of American basketball, teams are awarded three points (rather than the usual two) if they are able to score from a specified minimum distance. Naturally, three-point shots have lower probability of success than two-pointers, so teams are essentially choosing a risky portfolio when deciding the frequency with which to attempt shots from three-point range. Given the supposition that teams seek to maximize the probability of winning, this can be viewed as a threshold utility problem. The (unsurprising) observation that as the end of the game approaches, the losing team often increases the number three-point attempts (while the winning team sticks with a "safe" portfolio of two-point shots), is therefore analogous to the convexity results obtained above: the only thing teams care about is the "probability mass" that lies above the win/loss threshold; so the losing team chooses the high-risk strategy, because even though it might lower their *expected* final score, it increases the probability of a win. The step function underlying the team's objective effectively induces risk-seeking behavior below the threshold, and risk-averse behavior above it.[10] The problem we have formulated above is not limited to discrete outcomes with Bernoulli distributions, and we have in mind a less explicitly sought objective than winning a basketball game, but the logic underlying our result is the same: when the quality threshold is high, it is best to take the long-shot by specializing in a single good.

Our convexity results for Case 1 can be visualized in Figure 4, which shows the effect of consumption on threshold utility in outcome space. Consider, for example, the line $c_x x^{++} + c_y y^{++} = k$ (i.e., the line connecting the points $(0, \frac{k}{y^{++}})$ and $(\frac{k}{x^{++}}, 0)$). Given that the consumer chooses the consumption bundle (x^{++}, y^{++}), this line traces the outcomes (c_x, c_y) that result in the threshold level (k) of quality being exactly attained. Note that as drawn, $x^{++} = y^{++}$, and (x^{++}, y^{++}) lies in region A^{++}, because $\frac{k}{y^{++}} = \frac{k}{x^{++}} < 1$. Note also that because the random variables C_x and C_y are distributed uniformly (and independently) on the unit interval, probability mass for the joint distribution is (uniformly) distributed on the unit square in (c_x, c_y)-space. So given allocation (x^{++}, y^{++}), the area in the unit square above and to the right of the line $c_x x^{++} + c_y y^{++} = k$ is the probability of attaining k or more units of quality, and the area below and to the left of this line is the probability of attaining less than k units of quality. Now consider a slight shift in consumption, whereby the consumer gives up ε units of good x in exchange for $p\varepsilon$ units of good y. This results in a counter-clockwise rotation of the line $c_x x^{++} + c_y y^{++} = k$, so that the consumer "gains" area B^{++} while losing area L^{++}. It is easy to show that for $p \leq 1$, area B^{++} is smaller than area L^{++}, so the consumer would have been better off with the "diversified" bundle in which $x = y$.

It should be apparent from Figure 4 that this logic is reversed in region

[10] Rubin and Paul (1979) establish a similar result in the context of competition for mates in evolutionary history (which, they argue, may help explain the shift in risk attitudes typically observed between adolescence and adulthood among males).

A^{--}. In moving from allocation (x^{--}, y^{--}) (where $x^{--} = y^{--}$) to allocation $(x^{--} - \varepsilon, y^{--} + p\varepsilon)$, the consumer increases the probability (given $p \geq 1$) of exceeding the threshold. Hence for a sufficiently large threshold, preferences become non-convex.

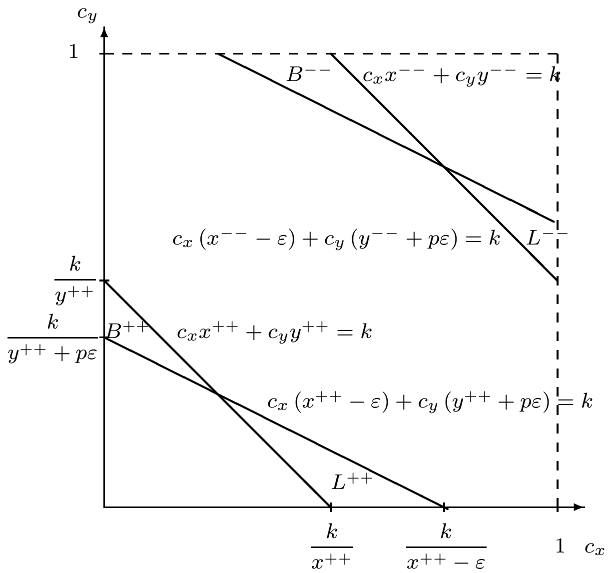

Figure 4: The Geometry of Threshold Utility

5 Generalization to Log-Concave Distributions

5.1 A General Statement

We now offer a generalization of the convexity properties and discontinuous threshold effect shown for Case 1 to random variables with *log-concave* density functions (i.e., random variables for which the logarithm of the density function is concave).[11] As might be expected, allowing for such a general class of random variables does not come without cost: several additional restrictions will be needed (the necessity of each will be shown in the next section), which underscore both the limitations of our framework and the importance of the product-specific probability distributions in determining threshold utility.

Definition 1 *If C_x and C_y are two random variables, then we say that C_x is* more peaked *than C_y, whenever*

$$P(|C_x - EC_x| \geq t) \leq P(|C_y - EC_y| \geq t)$$

[11] The uniform distribution, for example, is log-concave. See Bagnoli and Bergstrom (2005) for a review of economic applications of log-concavity.

for all $t \geq 0$. If the above inequality is strict for all $t > 0$ unless the probabilities are either both 0 or both 1, then we say that C_x is strictly more peaked *than C_y.*

We will employ Proschan's (1965) Lemma 2.1.

Lemma 1 *[Proschan 1965] Suppose that C_x and C_y are independent random variables both possessing a symmetric log-concave density function f. Then for any given $m > 0$ we have that $Z_{\lambda,m} := \lambda C_x + (m - \lambda) C_y$ is strictly increasing in peakedness in λ on $\left[0, \frac{m}{2}\right]$.*

It is also known that $Z_{\lambda,m}$ has a symmetric log-concave density function.

We investigate the following object function:

$$v(\lambda) = U(\lambda, m - \lambda) = P\left(Z_{\lambda,m} \geq k\right), \tag{4}$$

where $\lambda \in [0, m]$. We are now ready to state the following proposition:

Proposition 3 *Suppose that the independent nonnegative random variables C_x and C_y are symmetric around their common means $\mu = EC_x = EC_y$, have log-concave density f and supp(C_x) =supp(C_y) $= [0, 2\mu]$. Then we have the following four cases:*

1. *If $k < m\mu$, then v is strictly quasi-concave with a maximum at $\frac{m}{2}$ (interior solution).*
2. *If $m\mu < k < 2m\mu$, then v is strictly quasi-convex with two maxima at 0 and m (boundary solutions).*
3. *If $k = m\mu$, then v is constant.*
4. *If $2m\mu \leq k$, then v equals zero (death zone).*

Proof. Clearly, $EZ_{\lambda,m} = m\mu$, where $\lambda \in \left[0, \frac{m}{2}\right]$. Since f is symmetric around $m\mu$, it follows for all $t > 0$ and all $\lambda \in \left[0, \frac{m}{2}\right]$ that

$$\begin{aligned} P\left(m\mu - t \leq Z_{\lambda,m} \leq m\mu\right) &= P\left(m\mu \leq Z_{\lambda,m} \leq m\mu + t\right) \quad \text{and} &(5)\\ P\left(Z_{\lambda,m} \geq m\mu\right) &= \frac{1}{2}. &(6)\end{aligned}$$

Now pick two values α and β such that $0 \leq \alpha < \beta \leq m/2$. From Lemma 1 it follows that

$$P\left(|Z_{\beta,m} - m\mu| \leq t\right) > P\left(|Z_{\alpha,m} - m\mu| \leq t\right) \tag{7}$$

for all $m\mu > t > 0$. Combining (5) with (7), we obtain

$$P\left(m\mu - t \leq Z_{\beta,m} \leq m\mu\right) > P\left(m\mu - t \leq Z_{\alpha,m} \leq m\mu\right)$$

and therefore by (6) we get

$$P\left(m\mu - t \leq Z_{\beta,m}\right) > P\left(m\mu - t \leq Z_{\alpha,m}\right)$$

for all $m\mu > t > 0$. Finally, in case 1 by setting t equal to $m\mu - k$ we derive[12]

$$P\left(k \leq Z_{\beta,m}\right) > P\left(k \leq Z_{\alpha,m}\right).$$

Thus v is strictly increasing on $\left[0, \frac{m}{2}\right]$. In an analogous way one can establish that v is strictly decreasing on $\left[\frac{m}{2}, 1\right]$.

We derive the second statement from the first one. Assume that $m\mu < k < 2m\mu$. Then employing the symmetry of $C_x - \mu$ and $C_y - \mu$, we obtain

$$\begin{aligned}
v(\lambda) &= P\left(\lambda C_x + (m-\lambda)C_y \geq k\right) \\
&= P\left(\lambda(C_x - \mu) + (m-\lambda)(C_y - \mu) \geq k - m\mu\right) \\
&= P\left(\lambda(\mu - C_x) + (m-\lambda)(\mu - C_y) \leq m\mu - k\right) \\
&= P\left(\lambda(C_x - \mu) + (m-\lambda)(C_y - \mu) \leq m\mu - k\right) \\
&= 1 - P\left(\lambda(C_x - \mu) + (m-\lambda)(C_y - \mu) \geq m\mu - k\right) \\
&= 1 - P\left(\lambda C_x + (m-\lambda)C_y \geq 2m\mu - k\right) = 1 - v^*(\lambda),
\end{aligned}$$

where v^* stands for the (restricted) utility function in case of constraint $2m\mu - k > 0$. Since for the problem associated with v^* we have $E(\lambda C_x + (m-\lambda)C_y) = m\mu > k^* = 2m\mu - k$ by our assumption of case 2, v^* is strictly quasi-concave with minima at 0 and m by case 1; and therefore, v is strictly quasi-convex with maxima at 0 and m.

Case 3 follows by the continuity of u.

Finally, since C_x and C_y are nonatomic, nonnegative and symmetric with mean μ, the supports of C_x and C_y equal $[0, 2\mu]$, $\lambda C_x + (m-\lambda)C_y$ has to be smaller than $2m\mu$ with probability one, which completes the proof of our proposition. ∎

As a corollary of our previous proposition we obtain our main theorem.

Theorem 1 *[Discontinuous threshold effect] Under the Assumptions of our previous proposition, as k increases from 0 to $m\mu$ the optimal consumption bundle of the consumer remains $(\frac{m}{2}, \frac{m}{2})$, while from $m\mu$ to $2m\mu$ the two optimal consumption bundles are $(0, m)$ and $(m, 0)$. In particular, if the payoff threshold k increases from $m\mu - \varepsilon$ to $m\mu + \varepsilon$, where ε denotes a small positive value, then we observe a discontinuous shift in the consumer's behavior.*

5.2 Counterexamples: Continuous threshold effects

One important restriction in Theorem 1 is that we consider convex combinations of goods (i.e., we restrict our attention to budget lines with slope -1).[13] The following example shows that Theorem 1 cannot be generalized to arbitrary budget lines.

[12] Note that assumption $k < m\mu$ implies $t = m\mu - k > 0$.

[13] This restriction could be relaxed, of course, by transforming the distributions accordingly.

Case 2 *Arbitrary Price counterexample:*

$$f(x) = g(x) = \begin{cases} \frac{6}{7}\left(1 - x(x-1)\right), & \textit{if } x \in [0,1]; \\ 0, & \textit{if } x \notin [0,1]. \end{cases}$$

By solving Problem 1 for Case 2 we obtain utility function $U(x,y) =$

$$\begin{cases} 0 & \text{if } (x,y) \in A^0, \\ -\dfrac{\begin{array}{c} -60x^2yk^3 + 420x^2y^3k + 420x^3y^2k - 210xy^4k + \\ 210x^3yk^2 + 210xk^2y^3 - 210x^4yk - 15xyk^4 - \\ 60xy^2k^3 - 490x^3y^3 - 180x^2y^2k^2 + 34y^6 - 2k^6 + \\ 34x^6 + 69xy^5 - 210x^2y^4 - 210x^4y^2 + 69x^5y + \\ 6yk^5 + 6xk^5 + 30y^2k^4 + 30x^2k^4 - 140y^3k^3 - \\ 140x^3k^3 + 210x^4k^2 + 210k^2y^4 - 138x^5k - 138y^5k \end{array}}{490x^3y^3} & \text{if } (x,y) \in A^{--}, \\ \dfrac{\begin{array}{c} -210yk^2 - 420y^2k + 210xyk + 210xy^2 - 34x^3 - 69x^2y + \\ 490y^3 + 140k^3 - 210xk^2 + 138x^2k \end{array}}{490y^3} & \text{if } (x,y) \in A^{-+}, \\ \dfrac{\begin{array}{c} -210xk^2 - 420x^2k + 210xyk + 210x^2y - 34y^3 - 69xy^2 + \\ 490x^3 + 140k^3 - 210yk^2 + 138y^2k \end{array}}{490x^3} & \text{if } (x,y) \in A^{+-}, \\ \dfrac{\begin{array}{c} -60x^2yk^3 - 15xyk^4 - 60xy^2k^3 + 490x^3y^3 - \\ 180x^2y^2k^2 - 2k^6 + 6yk^5 + 6xk^5 + 30y^2k^4 + 30x^2k^4 \end{array}}{490x^3y^3} & \text{if } (x,y) \in A^{++}. \end{cases}$$

If we maximize U above the budget line $0.8x + y = 50$ and let the threshold vary from 24 to 25.5, we observe a continuous move from an interior solution to a corner solution. The optimal solutions as a function of k are depicted in Figure 5.

The next example shows that even for budget lines of slope -1 symmetry is a necessary assumption for establishing Theorem 1.

Case 3 *Non-symmetric counterexample:*

$$f(x) = g(x) = \begin{cases} 2x, & \textit{if } x \in [0,1]; \\ 0, & \textit{if } x \notin [0,1]. \end{cases}$$

Considering budget line $y = 50 - x$, the optimal values for x as a function of k can be found in Figure 6. We can observe a continuous move from the optimal interior solution $x^* = y^* = 25$ to the two corner solutions $x^* = 50, y^* = 0$ or $x^* = 0, y^* = 50$. Because of the symmetric setting we have two branches.

Figure 7 emphasizes, where $k = 32$, that even the peakedness statement of Lemma 1 is violated in Case 3.

Finally, we show that even log-concavity cannot be dropped without consequence in Theorem 1.

Case 4 *Non-log-concave counterexample:*

$$f(x) = g(x) = \begin{cases} 12\left(x - \frac{1}{2}\right)^2, & \textit{if } x \in [0,1]; \\ 0, & \textit{if } x \notin [0,1]. \end{cases}$$

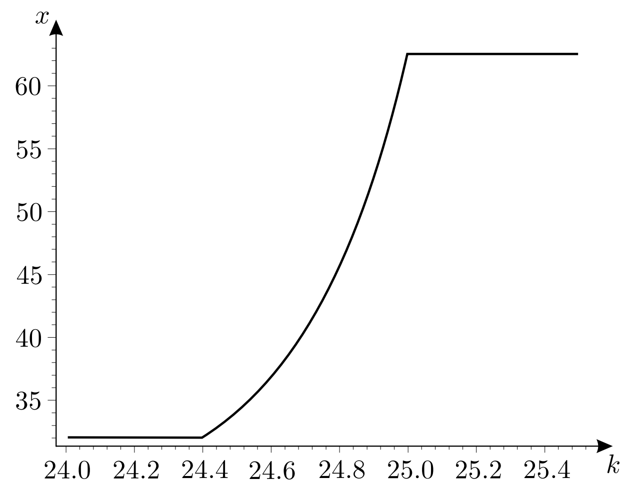

Figure 5: Case 2

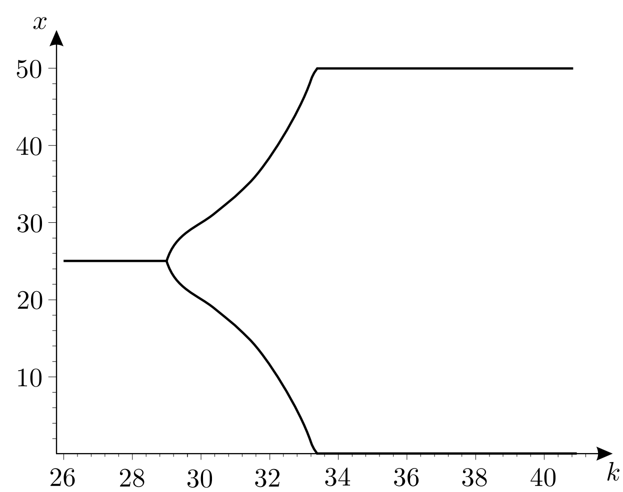

Figure 6: $x^*(k)$ for Case 3

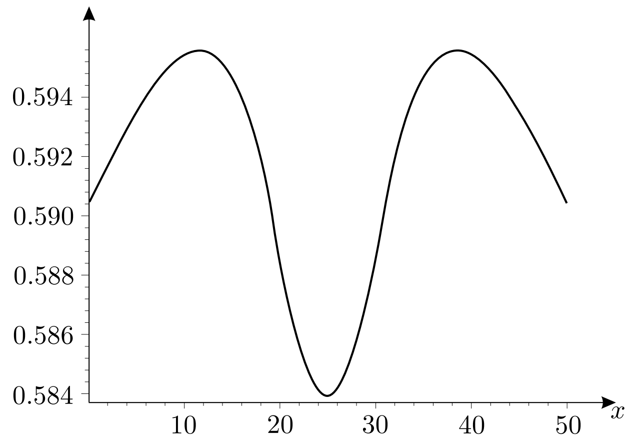

Figure 7: Violation of increasing peakedness by Case 3

Considering again budget line $y = 50 - x$, the optimal values for x as a function of k are shown in Figure 8. We can observe a continuous change in the consumer's behavior for values around $k = 10$. By the symmetric setting we have again two branches.

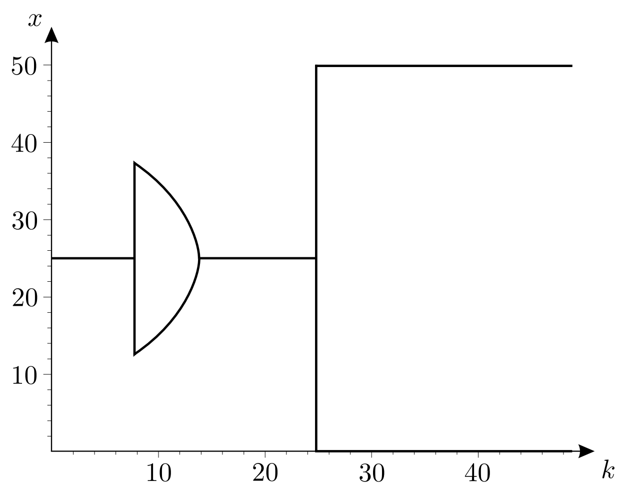

Figure 8: $x^*(k)$ for Case 4

Figure 9 emphasizes that even the peakedness statement of Lemma 1 is violated in Case 4. Figure 9 corresponds to $k = 10$.

Cases 2, 3, and 4 establish that Theorem 1 does not hold in general if the assumptions $p = 1$, symmetry, and/or log-concavity are violated.

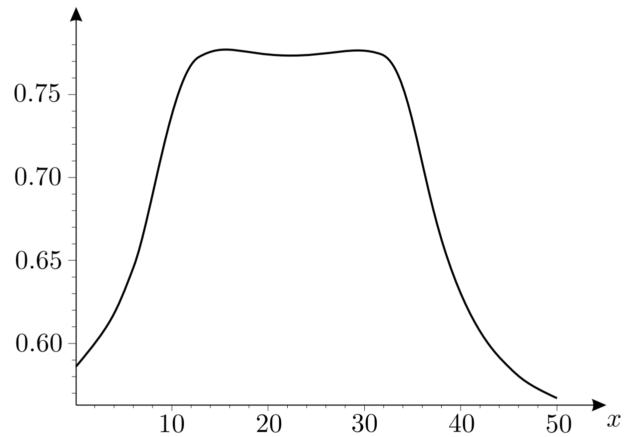

Figure 9: Violation of increasing peakedness by Case 4

5.2.1 Non-Identical Distributions

Next we consider the case of non-identical distributions.

Definition 2 *Random variable X dominates random variable Y in the sense of likelihood ratio, denoted by $X \succ Y$, if $f(x)/g(x)$ is non-decreasing in x, where f and g denote their respective density functions.*

We will employ Ma's (1998) Lemma 2.

Lemma 2 *[Ma 1998] Suppose that X and Y are independent random variables possessing symmetric log-concave density functions f and g, respectively. If $|X| \succ |Y|$, then for any given $m > 0$, $(m-\lambda)X + \lambda Y$ is strictly increasing in peakedness in λ on $\left[0, \frac{m}{2}\right]$.*

We investigate the following objective function:

$$v(\lambda) = U\left(\frac{m-\lambda}{p}, \lambda\right) = P\left(Z_{\lambda,m} \geq k\right), \tag{8}$$

where $\lambda \in [0, m]$ and $Z_{\lambda,m} := (m-\lambda)\frac{C_x}{p} + \lambda C_y$. We are now ready to state the following proposition:

Proposition 4 *Suppose that the independent nonnegative random variables C_x and C_y are symmetric around their means $p\mu = EC_x$ and $\mu = EC_y$, have log-concave densities f and g and $|C_x/p - \mu| \succ |C_y - \mu|$. If $k < m\mu$, then the consumer, facing budget line $px + y = m$ spends at least as much on good y as on good x.*

Proof. Clearly, $EZ_{\lambda,m} = m\mu$, where $\lambda \in \left[0, \frac{m}{2}\right]$. Since $Z_{\lambda,m}$ is symmetric around $m\mu$, it follows for all $t > 0$ and all $\lambda \in \left[0, \frac{m}{2}\right]$ that

$$P\left(m\mu - t \leq Z_{\lambda,m} \leq m\mu\right) = P\left(m\mu \leq Z_{\lambda,m} \leq m\mu + t\right) \quad \text{and} \tag{9}$$

$$P\left(Z_{\lambda,m} \geq m\mu\right) = \frac{1}{2}. \tag{10}$$

Now pick two values α and β such that $0 \leq \alpha < \beta \leq m/2$. From Lemma 2 it follows that

$$P\left(|Z_{\beta,m} - m\mu| \leq t\right) > P\left(|Z_{\alpha,m} - m\mu| \leq t\right) \tag{11}$$

for all $m\mu > t > 0$. Combining (9) with (11), we obtain

$$P\left(m\mu - t \leq Z_{\beta,m} \leq m\mu\right) > P\left(m\mu - t \leq Z_{\alpha,m} \leq m\mu\right)$$

and therefore by (10) we get

$$P\left(m\mu - t \leq Z_{\beta,m}\right) > P\left(m\mu - t \leq Z_{\alpha,m}\right)$$

for all $m\mu > t > 0$. Finally, in case 1 by setting t equal to $m\mu - k$ we derive[14]

$$P\left(k \leq Z_{\beta,m}\right) > P\left(k \leq Z_{\alpha,m}\right).$$

Thus v is strictly increasing on $\left[0, \frac{m}{2}\right]$, which implies that the consumer buys at most $m/(2p)$ units of good x and at least $m/2$ units of good y. ■

Proposition 4 implies that the consumer will not spend more on the "more risky" asset when the threshold is small. Note that under the assumptions of Proposition 4, the expected returns on a monetary unit of the two goods are identical; and that symmetry together with the likelihood dominance condition implies that the good with less spread should be preferred.

6 Economic behavior and threshold-induced non-convexities

It seems likely to us that threshold payoffs played an important role in the evolutionary history of the human species, and that this history is reflected in our collective psychological architecture. An obvious first example is the one with which we began this investigation: consumption of a sufficient quantity of limiting micronutrients. As noted in Smith (2004), food marketers appear to have stumbled on a deep truth about human nature: if you want to induce a dramatic upward shift in consumption of your product, broadcast messages proclaiming (at least implicitly) that this food has the power to cure illnesses of unknown origin.[15] In other words, provide information to the consumer

[14] Note that assumption $k < m\mu$ implies $t = m\mu - k > 0$.

[15] These messages, in which a "medical miracle" accompanies ingestion of the target product, are commonly employed in television advertisements for food aimed at children. In the pre-industrial world (in which the danger of illness or death by micronutrient deficiency was very real) such scenarios would have provided critical information about nutritional value (Smith 2002).

suggesting that a critical threshold is larger than might have otherwise been apparent, thus inducing a local nonconvexity that will drive up demand for your product.[16]

As noted by Rubin and Paul (1979) (see footnote 10), reproductive success represents an obvious source of threshold payoffs in human history. The ubiquitous use of sexual themes in commercial advertisements could, therefore be viewed as providing "information" (in the subconscious, or psychological sense of the word) to the consumer about a looming threshold: buy this product, increase the probability of winning an attractive mate.

Thresholds are also likely to be important outside the world of commercial marketing messages: consider the important–and ubiquitous–life decision about family size. A woman (and her mate) can choose to have many children–perhaps at the expense of per-child parental investment–or she can choose to have just one. Demographers have long known that fertility falls dramatically when economies transition from subsistence agriculture to an industrial economy (Jones *et al.* 1998). A concise explanation for this phenomenon might be due to the fact that in the developed world, the returns to education (i.e., parental investment) are much higher. In other words, the threshold for success in affluent societies is much higher, so parents respond by devoting more resources to fewer children.

There is also no reason our model could not be applied to choices made by producers. Diamond (2005) recounts the story of the Greenland Norse, who demonstrated a surprising reluctance to utilize new and innovative technologies in the production of food. Having immigrated some years before to a stark and desolate land, they faced a perennial looming threshold: growing enough food in the short summer season to survive the long, cold winters. Perhaps the risks of diversification (given the consequences of failure) made it optimal, in some sense, to maintain the old ways. More generally, some modern firms face the threat of bankruptcy–a fact of economic life that might push them toward extreme and high-risk (in other words, non-convex) business strategies.

As we noted at the outset, it is somewhat surprising that economics lacks a normative theory of convex preferences. It is our hope that in proposing one reason *why* preferences might be convex, we may have taken one small step toward being able to predict *when* preferences will be convex. And when they won't.

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[16]Interestingly, there is also some evidence that consumers who are under conditions of "stress" systematically choose less-balanced diets (Torres and Nowson 2007).

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