On convergence of contractual trajectories in pure exchange economies

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The study is aimed to investigate the convergence to equilibrium of trajectories generated by contractual processes. “Contractual process” is a view on economy’s self-regulation, describing any state of exchange economy as a set of barter contracts among agents/coalitions. It is assumed that any moment of continuous time agents can partially break some contracts and sign more beneficial ones. Such recontracting is called “benevolent,” when an agent/coalition breaks old contracts only when exhausting all other opportunities to increase welfare. Such processes are shown to converge to equilibrium under reasonable conditions, whereas non-benevolent processes need not converge, as shown by series of examples.

Keywords and Phrases: core, contract, contractual allocation, contractual process (trajectory), tâtonnement, competitive equilibrium.

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Non-Technical Summary

This study is purely theoretical, concerning our fundamental ideas about markets. We propose a new “contractual” approach describing economic behavior in disequilibrium situations. It develops a new version of market dynamics, adding one more approach to explain market functioning.

At least five different competing approaches have been elaborated so far to explain how real markets operate, and how economy attains equilibrium, namely: Walrasian tâtonnement, the disequilibrium dynamic model, Edgeworth’s processes, Smale-type processes and Strategic bargaining approach. Each has some advantages and shortcomings, but none is completely successful and convincing. The famous historian of economic thought, Mark Blaug commented on this puzzle in an interview for “Challenge” (May-June 1998). To the question “What are the major issues on which we have not made progress?” he replied: “Markets and how they actually function; that is, how they adjust to match demand and supply. We in economics know a hell of a lot about equilibrium, but we really don’t know how markets actually get to equilibrium.” We would add that, in reality, we hardly see equilibrium itself, but rather some infinite convergence process involving reaction to shocks. This research suggests a sixth approach to this puzzle.

The basic idea is that in disequilibrium situations, under imperfect information, an agent can adjust both prices and quantities to disparities in demand and supply. Moreover, she often trades with a limited group of other agents that can be looked upon as “coalition.” In different submarkets or coalitions, prices or “terms of trade” can be different, at least in disequilibrium. The notion of “contracts” enriches the classical concept of the “core” with dynamic details of making and breaking the personalized agreements and coalitions. The market process can then be looked upon as series of barter exchanges among emerging and declining agents’ groups. A new concept of contractual process and its related contractual trajectory express this idea. We study several reasonable types of contractual trajectories, corresponding to different behavioral hypotheses. Some of them do converge to Walrasian equilibrium.

Informally, a contractual process describes an infinite horizon, continuous time trajectory. In each time agents possess, trade and consume some renewable resources. At any moment, any agent can exchange goods within many groups of agents (coalitions): she does not belong entirely to any coalition. Any coalition has a contract, which is a plan of exchange. The summation of endowments and all these trading (barter) contracts is the (current) consumption bundle for the agent. In the next instant an agent can renew old contracts, or break some of them to look for a better ones. The procedure is repeated many times, and it may result in stable barter contracts. Though there are no money and no explicit prices, it turns out that this stable situation is a usual competitive equilibrium, with uniform rates of exchange (supporting market prices). The question is whether this equilibrium can be reached through a converging contractual process under reasonable assumptions. A positive answer would make the whole concept logically compelling, supplementing, or even replacing the classical views on “tâtonnement” and on the generation of market equilibrium.

The results of our analysis show that, under very general assumptions, the contractual processes may or may not converge to equilibria. A series of examples, where non-convergence is caused by different reasons, is provided. The most interesting positive results were obtained for so called “benevolent” processes that converge to equilibrium.
under reasonable assumptions. It is assumed that a coalition or an agent at first searches for new beneficial contracts among those offers that do not involve breaking other existing contracts. Only when that is impossible does the agent/coalition initiate new contracts, implying breaking, or decreasing in volume, old ones (i.e. the proportions of exchange remaining). This assumption requires rather benevolent and well-informed agents. However it is not too unrealistic to assume careful investigation of new possibilities before breaking the old contracts. It may also capture the impact of social institutions generating “trust” and “honesty” in business dealing.

There are some specific processes in microeconomics that appear similar in operation to this vision, namely “double auctions.” There are also trade environments, like stock and commodity exchanges, in which agents, as assumed here, adjust price and quantity simultaneously without an auctioneer. Each agent suggests a volume (lot) to sell or buy and its (ask/bid) price. These decentralized offers and contracts drive the market. Perhaps this contractual approach can help clarify our understanding of the dynamics of such systems.

Introduction

Modern models of economy, just as classical ones (Arrow-Debreu model or its simplest version, a pure exchange economy) are modelling processes of production and allocation of goods and are based on the concept of competitive equilibrium. At the same time, the proper mechanism of market functioning (how are prices settled or how do individuals, having chosen preferred consumption bundles, transit to final resource allocation?) still is not completely clear. In fact, classical presentation is that equilibrium prices are realized as a result of little by little permanently going tâtonnement process, which corrects current prices in accordance with excess demand law: price for a commodity increases if demand exceeds supply; when supply exceeds demand price decreases. Economic intuition says us that moving in this manner economic system as a whole has to find, to grope toward equilibrium prices. Applying mathematical terms this means that if one describes price change process by differential equation (inclusion), having in right hand side excess demand, then every solution of this equation converges to equilibrium prices. However what is this demand and can we observe it in reality? In mathematical model by definition demand is the summation of optimal individual solutions in consumer problems, which are defined by current non-equilibrium prices and by agents’ preferences. How is it possible to observe demand under non-equilibrium prices, if it is the sum of unrealized wishes to buy commodity bundles? One can observe the total volume of purchases or the volume of sellings, supply and its excess, but we think that demand is, evidently, fundamentally unobservable category.

Moreover, classical view on prices change in accordance with excess demand rule is commonly based on a fictitious auctioneer hypothesis. This auctioneer conducts prices, but he/she is not a revealed economic agent, more likely this is an impersonal being, realizing a market power.

In the modern literature are also available other approaches, different from classical

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1This result holds only under additional strong assumptions (gross substitutability and etc.), however up to this moment it is not important.

2If one knows supply and excess supply is positive, then demand for a commodity can be calculated, but what can be done when excess demand is positive?
tâtonnement, aimed on modelling of dynamics of market processes and the analysis of their convergence to equilibrium: processes of the prices changes, using Jacobi matrix of excess demand function (Smale’s approach and other, see [32], [33], [12], [13]); disequilibrium models of trade (Hahn’s process, Fisher’s approach and other, [10], [9], [34] [4], [24]–[26]); Edgeworth’s processes [9], [28], [7], [18], [8], [6] etc., see bibliography. One can find an extensive review of the literature on this theme in Appendix. However all approaches have the shortcomings, partially the same as in Walrasian tâtonnement (auctioneer and other), partially new, as, for example, high information requirement of processes with Jacobian and others. So, classical and other modern views on the market and laws of its homeostasis are not quite satisfactory from the modern point of view.

However can we suggest a constructive idea to solve the problem of attainability of economic equilibria and for better understanding of dynamics of market processes? To answer these questions we seemingly need to reconsider our views of what really occurs in the market. In our opinion there are a lot of commodity exchange dealings and for all involved individuals these dealings are mutually beneficial at the moment of their realization. The current resource allocation is generated as a summation of all the accomplished dealings and of an initial endowments allocation. During a time some new dealings are realized, some of them reiterate there made earlier, other ones do not (this can be treated as a form of the rejection of signed in the past dealings which are non-beneficial at the moment). It is extremely important, that such “natural” process of a barter exchange goes itself, there is not here presented neither demand with the supply, nor prices. The investigation is aimed to formally describe and to study properties of these processes. Idea of the barter bargain by no means new in theoretical economics (e.g., in Edgeworth’s views), but it usually appeared as an interpretation, in the form of net trades in a formal model. Problem however namely in adequate formally-mathematical description of barter process, allowing an opportunity of a refusal from the bargains (breaking of the contract).

For the formalization of suggested point on market functioning we propose to apply contract based approach. The first attempts to introduce the notion of contract in exchange economies were made by Makarov [19], [20] and Kozyrev [14], [15], which offered the idea of partial breaking of the contracts. Hereinafter the theory of the contracts was essentially reconsidered and advanced in [21]. Contractual approach is closer to an intuitive imagination on real processes of forming of prices and consumed resources and to deliver better understanding of cooperative and individual features of agents’ behavior in a market. In particular, applying contractual approach one can suggest the clearer description of transition processes to stable (non-dominated) allocations and reveal a specific cooperative tâtonnement process, which formally-mathematical description is one of goals of this study. This cooperative tâtonnement supposes that coalitions of agents are able to sign new mutually beneficial contracts (exchange commodity dealings) and also each agent can partially break contracts signed in the past if it is beneficial for him. The process of the signing of new and breaking of old contracts is going in simultaneous mode and is extended over time. The last means, that in fact process deals with momentary contracts, which together with signed in the past define the process derivative.

Formally contractual process can be described via differential inclusion \( \dot{x}(t) \in F(x) \), where \( x(t) \) is current allocation of resources. The right hand side of this inclusion is formed via mutually beneficial contracts for various coalitions where abilities of singleton coalitions are realized by means of partial breaking of contracts. All feasible solutions of this inclusion form a set of feasible contractual trajectories, which can or cannot converge.
to (potentially) final allocations. It is known from the theory of contracts [21] that under some assumptions (interior point, differentiable concave utilities) every proper contractual allocation (this is an allocation which can be realized by a web of contracts stable relative to the signing of new contracts and relative to partial breaking of old ones) is equilibrium allocation. The converse implication is always true: every equilibrium can be presented as a proper contractual allocation. Thus equilibria and only they (under assumptions) are stationary points for cooperative proper contractual tâtonnement. However up to this moment it is unknown when this process, starting at initial endowments, is converging and which stationary points are stable. Exactly the investigation of convergency of contractual processes and related questions is the main goal of this paper.

1 Dynamical contractual process as a cooperative tâtonnement

We begin with description of contractual economy and main contract based concepts in an appropriate form of generality. Further the basis of dynamical contractual processes is described.

1.1 Main concepts of contractual economy

Let us consider a typical exchange economy in which \( E = \mathbb{R}^l \) denotes the space of commodities (\( l \) is the number of commodities). Let \( I = \{1, \ldots, n\} \) be a set of agents (traders or consumers). A consumer \( i \in I \) is characterized by a consumption set \( X_i = E_+ = \mathbb{R}^l_+ \), an initial endowments \( \omega_i \in X_i \), and a preference relation described by a utility function \( u_i : X_i \to \mathbb{R} \), where \( u_i(x_i) > u_i(y_i) \) means that agent \( i \) strictly prefers a bundle \( x_i \) to \( y_i \). This may be also standardly denoted as \( x_i \succ_i y_i \). So, the pure exchange model under study may be represented as a triplet:

\[
E = (I, E, (X_i, u_i(\cdot), \omega_i)_{i \in I}).
\]

A pair of vectors \((x, p), x = (x_1, \ldots, x_n) \in \prod_i X_i, p \in \mathbb{R}^l\), is said to be a Walrasian or competitive equilibrium of model \( E \), if the price vector \( p \neq 0 \) and the following conditions are satisfied:

(i) \( \forall i \in I, \; px_i \leq p\omega_i \; \& \; \forall y_i \in X_i, \; y_i \succ_i x_i \Rightarrow py_i > p\omega_i \);

(ii) \( \sum_{i \in I} x_i = \sum_{i \in I} \omega_i \).

Let us denote by \( L = E^n \) the space of economy allocations, let \( \omega = (\omega_i)_{i \in I} \) be the vector of initial endowments of all traders of the economy. Denote \( X = \prod_{i \in I} X_i \) and define

\[
A(X) = \{x = (x_i)_{i \in I} \in X \mid \sum_{i \in I} x_i = \sum_{i \in I} \omega_i\},
\]

the set of all feasible allocations in \( E \).

Everywhere below we shall assume that model \( E \) satisfies the following smoothness assumption (S).
(S) All utilities $u_i(\cdot)$ are concave and twice continuously differentiable functions, such that $\forall x_i \in X_i = \mathbb{R}_+^I$, $\forall i \in I$, $\nabla u_i(x_i) \neq 0$ and matrices $\nabla^2 u_i(x_i)$ are negative definite.

Further we shortly recall different contractual concepts, see [21].

By the formal definition, any reallocation of commodities $v = (v_i)_{i \in I} \in L$, where $v_i \in E$, $i \in I$, i.e., any vector $v \in L$ satisfying $\sum v_i = 0$, is called a (barter) contract. In this project context we assume that every contract is permissible.

A finite collection $V$ of permissible contracts is called a web of contracts relative to $y \in \mathcal{A}(X)$ if

$$y + \sum_{v \in U} v \in X \quad \forall U \subset V.$$ 

A web of contracts $V$ relative to $\omega$ is called a web of contracts or simply a web. Note that $V = \emptyset$ is a web relative to every $y \in \mathcal{A}(X)$. Notation $x(V) = \omega + \sum_{v \in V} v$ denotes the feasible allocation sustained by $V$ relative to $\omega$. For any contract $v \in V$, let us set

$$S(v) = \text{supp}(v) = \{i \in I | v_i \neq 0\},$$ 

the support of the contract $v$. It is assumed that contract $v \in V$ may be broken by any trader in $S(v)$, since he/she simply may not keep his/her contractual obligations. Also a non-empty group (coalition) of consumers can sign any number of new contracts. Being applied jointly, i.e., as a simultaneous procedure, these operations allow coalition $T \subseteq I$ to yield new webs of contracts. The set of all such webs is denoted by $F(V, T)$. It is required formally that each element $U \in F(V, T)$ has to satisfy the following properties:

(i) $v \in V \setminus U \Rightarrow S(v) \cap T \neq \emptyset$,

(ii) $v \in U \setminus V \Rightarrow S(v) \subseteq T$.

Condition (i) means that only members of $T$ can break contracts in $V$, condition (ii) means that only members of $T$ may sign new contracts.

In contract-based approach the notion of domination via a coalition is extended onto webs of contracts. This property of domination via coalition $T \subseteq I$, being written as $U \succ_T V$ ($U$ dominates $V$ via coalition $T$), means that

(i) $U \in F(V, T)$,

(ii) $x_i(U) \succ_i x_i(V)$ for all $i \in T$.

A web of contracts $V$ is called stable if there is no web $U$ and no coalition $T \subseteq I$, $T \neq \emptyset$ such that $U \succ_T V$.

A web of contracts $V$ is called lower stable if there is no web $U$ and no coalition $T \subseteq I$, $T \neq \emptyset$ such that $U \succ_T V$ and $U \subseteq V$.

A web of contracts $V$ is called upper stable if there is no web $U$ and no coalition $T \subseteq I$, $T \neq \emptyset$ such that $U \succ_T V$ and $V \subseteq U$.

An allocation $x$ is called contractual (lower, upper contractual) if $x = x(V)$ for a stable (lower, upper stable) web $V$. 

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It can be directly deduce from definitions that in any standard market every core allocation allows an alternative description as contractual one; accordingly, Pareto optimal allocations correspond to upper contractual ones, and individual rational allocations are lower contractual ones etc. The concept of proper contractual allocation is also important, this concept realizes (assumptions: interior point, smooth preferences) an alternative description of equilibria.

The notion of proper contractual allocation is introduced due to the following construction. First let us introduce an equivalence relation on the set of all lower stable webs, this equivalence will allow us to partially divide contracts. To this end, let us define a partial ordering on the set of all webs as follows:

\[ U \geq V \iff \exists \text{ a map onto } f : U \to V, \text{ such that} \]

(i) \( \lambda f(u) = u \) for some \( 0 \leq \lambda \leq 1 \) and for every \( u \in U \),

(ii) \( \sum_{u \in f^{-1}(v)} u = v \) for every \( v \in V \).

Thus, a relation \( U \geq V \) simply means, that contracts from \( U \) are produced from the contracts from \( V \) due to partition into several contracts (decomposition in a sum) under condition of preservation of exchange proportions and volumes of exchanged commodities.³

Now the equivalence relation may be defined as follows:

\[ U \simeq V \iff \exists \text{ a web } W \text{ such that } V \geq W \& U \geq W. \]

**Definition 1.1** An allocation \( x \) is called proper contractual if there exists a web \( V \) such that \( x = x(V) \) and for every \( U \simeq V \) the allocation \( x = x(U) \) is contractual.

The economic meaning of proper contractual stability of an allocation is, that we allow the agents not only to sign new contracts but also to partially break contracts if exchange proportions remain constant. This extends agents’ operating potentialities and approaches contractual processes to market processes under perfect competition conditions. It is easy to see (e.g. see [21]) that under some technical assumptions all proper contractual allocations are equilibria.

Further let us turn to the main subject of the study. We suggest to investigate the stability of trajectories which correspond to the proper contractual behavior of traders. However before we would like to specify one possible interpretation of proper-contractual behavior, driving economy to proper-contractual allocations.

Suppose that an economy is not static and lives during long-duration interval of time. As time elapsed individuals sign the rather short-term contracts on an exchange of commodities. The contract assumes mutual deliveries of goods among agents and, after its execution, an opportunity of renewal, i.e., the same contract can be signed again, but now it is realized during another time period. The agents can agree with contract’s re-newal (prolongation) or disagree, first studying an opportunity to prolong contract in smaller volumes. Thus, instead of breaking of the contract, even if partial, for economy in dynamics living a long time period one can speak about renewal and non-renewal of the contracts. Notice, that if resources are renewed then according to this interpretation

³A value \( f(u) \in V \) specifies the contract, a share of which is \( u \in U \).
agents can consume goods as time goes on, notwithstanding the fact that current situation is a disequilibrium one. It seems natural to assume that stable in time contracts, i.e., regularly renewed contracts have to take out economy to equilibrium performance (there is no production!). However the convergence to such state is not clear and requires the careful research.

1.2 On the definition of contractual trajectory

Formally, a trajectory is a map \( x(\cdot) \), operating from \([0, +\infty)\) into the set of all feasible allocations, i.e., into \( \mathcal{A}(X) \),

\[
x(\cdot) : [0, +\infty) \to \mathcal{A}(X).
\]

Here the vector \( x(t) = (x_i(t))_2 \) is a feasible bundle of consumption plans, realized at the moment \( t \geq 0 \). It is presumed that \( t = 0 \) is the initial time point, the process ‘starts’ at this point from initial endowments allocation, i.e., we set \( x(0) = \omega \).

We are interested in not arbitrary trajectories of this type, but trajectories which can be realized during contractual processes via commodity exchange among agents. Presume \( \Delta t > 0 \) is the time period during which a contract \( v \) is realized, and presume that other exchange operations with commodities (the signing of new contracts or the breaking of existing ones) were not realized. Then at the moment \( t' = t + \Delta t \) trajectory takes value \( x(t') = x(t) + v \), wherefore \( v = x(t') - x(t) \). As soon as other contractual operations in interval \([t, t']\) were not conducted, one may think that at the point \( t'' = \lambda t' + (1 - \lambda)t \), \( \lambda \in [0, 1] \) the trajectory value is produced from values of end points, which are mixed in the same proportions, i.e., one can postulate \( x(t'') = \lambda x(t') + (1 - \lambda)x(t) \). This can be rewritten in the form \( x(t'') = x(t) + \lambda v \Rightarrow x(t + \lambda \Delta t) - x(t) = \lambda v \) and therefore,

\[
\dot{x}(t) = \lim_{\lambda \to 0} \frac{x(t + \lambda \Delta t) - x(t)}{\lambda \Delta t} = \frac{v}{\Delta t} \implies v = \dot{x}(t)\Delta t.
\]

Further let us assume that during time interval \([t, t']\) there were a (finite) sequence of signed contracts, such that their time periods of realization are not overlapping. Let \( m \) be a number of contracts. One can think that the final time point of one contract is simultaneously the starting point of another contract: if not we can always replenish system with an appropriate number of zero contracts. So, interval \([t, t']\) is divided into \( m \) intervals, determined by points \( t = t_0 < t_1 < \ldots < t_m = t' \), such that \([t_{k-1}, t_k]\) are time periods of contracts \( v_k = x(t_k) - x(t_{k-1}) \) realization, \( k = 1, \ldots, m \). Put \( \Delta t_k = t_k - t_{k-1} \) and due to previous formula find

\[
x(t') = x(t) + \sum_{k=1}^{m} v_k = x(t) + \sum_{k=1}^{m} \dot{x}(t_{k-1})\Delta t_k = x(t) + \int_t^{t'} \dot{x}(s)ds.
\]

As soon as by assumption contractual process starts at the moment \( t = 0 \) at the point \( \omega \), we have

\[
x(t) = \omega + \int_0^t \dot{x}(s)ds, \quad \dot{x}(s) = \frac{v_k}{\Delta t_k}, \quad \forall s \in [t_{k-1}, t_k].
\]

Further holding away ourself from the latter (simple) deduction or in other words, if we allow ourself to consider a limit variant of last formula then the number of contacts is

\footnote{Applying ‘physical’ interpretation, one can say that we postulate the uniform (constant) speed of contract realization in interval \([t, t']\).}
passing to infinite and the realization time of each contract is passing to zero, one can do
the following conclusions.

(i) **Contractual trajectory**, which for a finite number of contracts is represented as
integral of some step function, in general case is the integral of some integrable
the following conclusions.

\[ x(t) = \omega + \int_0^t \dot{x}(s)ds. \quad (1.1) \]

So, **contractual trajectory** is an absolutely continuous\(^5\) on every interval \([0, t], t > 0\) map
\(x(\cdot) : [0, +\infty) \to \mathcal{A}(X)\).

(ii) **Derivative** \(\dot{x}(\cdot)\) of contractual trajectory in general case is defined almost every-
where on \([0, +\infty)\) and the value \(\dot{x}(t)\) defines a (momentary) contract, signed at the mo-
ment \(t \in [0, +\infty)\). If the time \(\Delta t > 0\) of contract realization is known, that formally
means \(\dot{x}(t') = \dot{x}(t''), \forall t', t'' \in [t, t + \Delta t]\), the resulting (gross) contract can be found from
\(v(t) = \dot{x}(t)\Delta t\). In other words, the derivative of contractual trajectory can be under-
stood as a barter contract per time unit. Notice also the obvious corollary: the range
of derivative is the subspace of contracts, i.e.,

\[ \dot{x}(\cdot) : [0, +\infty) \to L^c, \quad L^c = \{ v \in L \mid v = (v_i)_I : \sum_i v_i = 0 \}. \quad (1.2) \]

One more remark in addition. What is a contract for the trajectory? By trajectory
definition we can not determine it because in general we do not know the duration of
contract’s realization (what does mean zero duration?). This is why one can correctly say
only about momentary contracts, or about the summation of contracts, signed during a
non-zero time interval.

Surely, items (i), (ii) do not describe all properties of contractual trajectory related
with contractual processes; these are only initial, unconditional requirements. In addition
it is necessary to take into account conditions, at which contracts are signed, and also
character of a trajectory changes under the breaking of contracts.

For the constructive description of contractual processes, related with the breaking
of contracts, it is convenient to consider extended understanding of a trajectory that we
shall call this a **coalitional** trajectory.

Suppose that for each coalition \(S \subseteq I\) with at least two elements, \(\text{card}(S) \geq 2\), an
(absolutely continuous) map

\[ v^S : [0, +\infty) \to L^c_S, \quad L^c_S = \{ v \in L \mid v = (v_i)_I : \sum_{i \in S} v_i = 0 \& v_i = 0, \forall i \notin S \} \quad (1.3) \]

is determined. Essentially, \(v^S(t)\) is gross (total) contract, achieved by the members of a
coalition \(S\) at a moment \(t \geq 0\). A collection of all such maps \(\{v^S(t)\}_{S \in \mathbf{K}} = V(t)\), related
with a set of permissible coalitions \(\mathbf{K} \subseteq 2^I\), obviously determines a trajectory in previous
sense by formula

\[ x(t) = \omega + \sum_{S \in \mathbf{K}} v^S(t), \quad t \geq 0. \quad (1.4) \]

\(^5\)A function \(f(\cdot)\) with domain \([a, b]\) is said to be absolutely continuous if \(\forall \varepsilon > 0 \ \exists \delta > 0\) such that
\(\sum_{k=1}^m |f(b_k) - f(a_k)| < \varepsilon\) holds for every finite system of pairwise non-overlapping intervals \((a_k, b_k) \subseteq (a, b),
k = 1, 2, \ldots, m\), which obeys \(\sum_{k=1}^m (b_k - a_k) < \delta\).
This presentation of contractual trajectory actually describes not only a current allocation, but a set of varying with time contracts, where each coalition has the only current gross contract (for forbidden coalitions — zero). As time elapsed this set can be transformed according to the rules of proper-contractual behavior. Therefore, to ensure that an allocation realized after partial breaking of the contracts from \( V(t) \) is feasible, it is necessary in addition to require that \( V(t) \) is a web of contracts (relative to \( \omega \)). Notice that coalitional-contractual trajectory is not a new object, but just a convenient form for representation of information in an adequate aggregated kind.

### 1.3 Contracting and recontracting processes

A coalition can sign a new contract only if all members of coalition have relevant motives in signing, i.e., after contract’s realization (up to current moment) the utility of every member has to increase. In last section we have seen that contract per time unit is the derivative of trajectory at time point. Thus for smooth preferences one can think that contract \( v \) will be signed by coalition \( S \), i.e., trajectory moves along vector \( v = \dot{x}(t) \) only if supp \( (v) = S \) and

\[
\langle \dot{x}_i(t), \nabla u_i(x(t)) \rangle > 0, \quad \forall i \in S.
\]

Since \( v_i = 0 \) if \( i \notin S \), then we can write a determining condition:

\[
\dot{x}_i(t) \neq 0 \Rightarrow \langle \dot{x}_i(t), \nabla u_i(x(t)) \rangle \geq 0, \quad \forall i \in I, \quad \forall t \geq 0. \tag{1.5}
\]

This condition characterizes moment \( t \) as the case of contract’s signing. Now let us consider the case of contracts’ breaking.

The description of contractual process with the partial breaking of contracts is possible in rather general framework. However in such a case the formal-mathematical analysis of process looks very difficult. This is why further we shall make several simplifying hypotheses. These hypotheses determine basic parameters: which contracts, in which time moment and in which volume are broken off, i.e., all vagueness of contractual process related with the breaking of contracts are revealed.

The decision on partial break of the contracts is accepted by each agent individually, in conditions of a sufficient information for myopic-rational breaking of the contracts. We conceive that, in difference with a signing of the new contract, where an individual needs to find the partners and to pass a stage of negotiations about the future contract, the breaking of contracts is simpler decision and, therefore, can be accepted and is realized without temporary delays, as soon as there is the suitable opportunity. This motivates the following hypothesis.

**(IB)** *Instantaneous Breaking of the contracts.* At every time moment each individual instantly (for zero time) partially breaks the signed earlier contracts in an optimum volume.

This hypothesis does not say anything about what contracts and in which volume can be broken off. In a general case pertinently to think, that each individual has an opportunity (right) to break in any volume any contract, signed earlier current time moment \( t \geq 0 \). However, to simplify the subsequent analysis, it is possible to consider some particular cases, a little bit limiting opportunities on break of the contracts.

For the aggregated contractual trajectory, defined in (1.1), (1.2) we shall postulate:
(UB) **Uniform Breaking of all contracts.** At every time moment each individual can partially break all contracts, signed in economy to the given moment, but just in identical measure (proportion).

This hypothesis assumes, that at a current moment \( t \in [0, +\infty) \) each agent makes a decision on break of the contracts. This decision is based on minimum of the information, extracted only from current allocation \( x(t) \) and not accepting in attention the values \( x(t') \), “passed” by a trajectory in previous time moments \( t' \in (0, t) \). Apparently, such sight on an opportunity to break contracts is acceptable for an economy with small number of agents, where it is possible to assume, that the contracts are signed only by a coalition of all agents. However if economy consists of many agents this assumption is problematic. Really, it is not clear why do effect of breaking of contracts with involved persons has to influence in such crucial manner — the break in the same measure — on uninvolved directly into contract individuals? However, to carry out break only for a part of the contracts in which agent is involved, that is better for essence of contractual process, it is necessary that this part to be explicit. A simple variant of revealing this information is to consider the trajectory in coalitional-contractual form described in the previous section.

For coalitional trajectory, described in (1.3), ( 1.4) we shall assume:

(CUB) **Coalitional Uniform Breaking of contracts.** At every moment of time each individual can partially break all contracts, signed by any coalition, in which she/he participates, and in limits of a coalition in an identical measure, but, probably, in different proportions for different coalitions.

From the informational point of view this hypothesis means, that each agent stores (remembers) the aggregated information about intra-coalition exchanges, in the form of “gross” contract. Thus, now the breaking of contracts by an individual influences only the agents directly involved in the barter contract with this individual by means of gross coalitional contract, and it does not concern to exchanges in other coalitions.

We are not studying coalition contractual trajectories in this paper. This is because of volume constrains and mainly because the analysis is quite difficult. Really, it is possible to describe processes of this type however the study of convergency is confronted with serious mathematical difficulties and up to now it is not clear when convergency takes place. More details and examples one can find in [22].

2 **UBI-processes: uniform breaking of contracts**

In this section we consider the case of aggregated proper-contractual trajectory, for which it is admitted partial breaking of all contracts signed to the current moment, and all in an equal measure.

Further we have to clarify that means the fact that an individual \( i \) at the moment \( t \) did not want, but after contract \( v \) signing, at some moment \( \tau > t \) he/she wants to partially break contracts. The first means that \( \langle \nabla u_i(x_i(t)), x_i(t) - \omega_i \rangle \geq 0 \), the second one that \( \langle \nabla u_i(x_i(t) + (\tau - t)v_i), x_i(t) + (\tau - t)v_i - \omega_i \rangle < 0 \). Therefore, does exist a moment \( t + \Delta t \in [t, \tau] \) such that

\[
\langle \nabla u_i(x_i(t) + \Delta t v_i), x_i(t) + \Delta t v_i - \omega_i \rangle = 0.
\]
Notice also, that by virtue of (IB) the effect of contracts’ breaking can influence the change of a trajectory at a moment \( t \) if and only if in each neighborhood of \( t \) there is a moment \( \tau > t \) with the specified properties. Passing \( \tau \to t \) we obtain

\[
\langle \nabla u_i(x_i(t)), x_i(t) - \omega_i \rangle = 0,
\]

and this is the first condition, which defines the motion of trajectory with contracts’ breaking. Notice that the point defined by equation (2.1) is the maximal point of utility \( u_i(y_i) \) on the ray starting at the point \( \omega_i \) in direction \( x_i - \omega_i \) (here \( y_i = \omega_i + \lambda(x_i - \omega_i) \), \( \lambda \geq 0 \)).

Further let us consider another condition. Primary, at the time \( t \) of signing, contract \( v \) was mutually beneficial at the point \( x(t) \). The fact that at the moment \( \tau > t \) an individual \( i \in \text{supp}(v) \) partially breaks gross contract \( x(t) + (\tau - t)v - \omega \) in a volume \( 1 - \alpha \) means that from the point \( x(t) \) the trajectory moves to point \( z = \omega + \alpha(x(t) + (\tau - t)v - \omega) = (1 - \alpha)\omega + \alpha(x(t) + (\tau - t)v), \ 0 \leq \alpha < 1 \). Once again this is a maximum point for agent \( i \)'s utility on the linear segment, linking \( x(t) + (\tau - t)v \) with initial endowments vector (it is a projection along straight line going through two points). Therefore new point of trajectory has to satisfy

\[
\langle \nabla u_i(x_i(\tau)), x_i(\tau) - \omega_i \rangle = 0.
\]

Setting \( \Delta t = \tau - t \) and substituting in equation expressions

\[
x_i(t + \Delta t) = x_i(t) + \Delta t \dot{x}_i(t) + o(\Delta t),
\]

\[
\nabla u_i(x_i(t + \Delta t)) = \nabla u_i(x_i(t)) + \nabla^2 u_i(x_i(t))(\Delta t \dot{x}_i(t) + o(\Delta t)) + o(\Delta t \dot{x}_i(t) + o(\Delta t)),
\]

which are true due to Taylor’s formula,\(^6\) and taking into account (2.1), we find

\[
\Delta t \langle \nabla u_i(x_i(t)), \dot{x}_i(t) \rangle + \Delta t (\nabla^2 u_i(x_i(t)) \dot{x}_i(t), x_i(t) - \omega_i) + \\
\Delta t^2 (\nabla^2 u_i(x_i(t)) \ddot{x}_i(t), \dot{x}_i(t)) + o(\Delta t) = 0.
\]

Dividing this on \( \Delta t \) and passing to limit over \( \Delta t \to 0 \) we are coming to equation

\[
\nabla u_i(x_i(t)) \ddot{x}_i(t) + \langle \nabla^2 u_i(x_i(t)) \dot{x}_i(t), x_i(t) - \omega_i \rangle = 0 \iff \\
\langle h_i(x_i(t)), \dot{x}_i(t) \rangle = 0, \quad h_i(x_i(t)) = \nabla u_i(x_i(t)) + \nabla^2 u_i(x_i(t))(x_i(t) - \omega_i).
\]

Equations (2.1), (2.2) describe important properties of contractual trajectory but still do not completely define process. It is necessary also take into account the dependence of \( \dot{x}(t) \) from initially mutually beneficial contract \( v \), signing which an agent comes to beneficial break of contracts. This is illustrated in Figure 1 (in Edgeworth box style), which reflects a character of transition and objects involved into analysis.

Recall that from the point \( x(t) \) trajectory moves on to the point \( x(t + \Delta t) = \omega + \alpha_i(x(t) + \Delta tv - \omega), \ 0 \leq \alpha_i < 1 \) at the moment \( t + \Delta t \). In general the value \( \alpha_i \) depends on current consumption \( x_i(t) \), (momentary) contract \( v \) and duration \( \Delta t > 0 \) of its realization. By virtue of (S) the model is smooth, and it is easily to see, that \( \alpha_i(x, v, \Delta t) \) is (locally) differentiable function, implicitly defined by equation

\[
\langle \nabla u_i(x_i(t + \Delta t)), x_i(t + \Delta t) - \omega_i \rangle = 0, \quad x_i(t + \Delta t) = (1 - \alpha_i)\omega_i + \alpha_i(x_i(t) + \Delta tv_i).
\]

\(^{6}\) \( o(\cdot) \) is the standard notation of infinitesimal value.
Here parameter $\alpha_i \geq 0$ determines a point $(1 - \alpha_i)\omega_i + \alpha_i(x_i(t) + \Delta tv_i)$ of $i$’s utility maximum on the line, parameterized as: $\omega_i + \lambda(x_i(t) + \Delta tv_i - \omega_i)$, $\lambda \geq 0$. If $\alpha_i < 1$, then at a point $x_i(t) + \Delta tv_i$ the breaking of contracts is realized in volume $1 - \alpha_i$, and, there is no the break if $\alpha_i \geq 1$.

From representation $x_i(t + \Delta t)$ in the right part of (2.3) we have
\[
\frac{x_i(t + \Delta t) - x_i(t)}{\Delta t} = \frac{(\alpha_i(x, v, \Delta t) - 1)}{\Delta t}(x_i(t) - \omega_i) + \alpha_i(x, v, \Delta t)\nu_i,
\]
whence, passing to a limit over $\Delta t \to 0$, with the account $\alpha_i(x(t), v, \Delta t) = 1$ (by virtue of (2.1)), we obtain
\[
\dot{x}_i(t) = \lambda_i(x_i(t) - \omega_i) + v_i, \quad \lambda_i = \frac{\partial \alpha_i(x(t), v, \Delta t)}{\partial \Delta t}|_{\Delta t=0}.
\]

Further, value $\lambda_i$ is possible to find from the equation (2.2),
\[
\langle h_i(x_i(t)), \lambda_i(x_i(t) - \omega_i) + v_i \rangle = 0 \Rightarrow \lambda_i = \frac{\langle h_i(x_i(t)), v_i \rangle}{\langle h_i(x_i(t)), (\omega_i - x_i(t)) \rangle}. \tag{2.4}
\]
Thus, if at a moment $t$ there exists only one agent, satisfying to (2.1), with number $i$, the trajectory locally obeys the law
\[
\dot{x}(t) = \lambda_i(x, v)(x(t) - \omega) + v, \quad \lambda_i(x, v) = \frac{\langle h_i(x_i(t)), v_i \rangle}{\langle h_i(x_i(t)), (\omega_i - x_i(t)) \rangle}.
\]
Moreover, presented considerations allow to reveal complete conditions, describing a moment $t$ as a situation of break of the contracts at a current allocation $x(t)$ and when a (momentary) contract $v$ is signing. The break is realized if the individual $i$ satisfies (2.1), value $\alpha_i(x(t), v, \Delta t)$ is locally decreased in $\Delta t$ at a point $\Delta t = 0$, i.e., if derivative with respect to $\Delta t$ is negative. So, for breaking it is necessary and sufficient that $\lambda_i(x, v) < 0$.

---

7When $\lambda_i(x, v) = 0$ the point $x = x(t)$ can be or not to be a limit point of breaking contracts points of a trajectory.
Further, by virtue of the assumption (S) the matrix $\nabla^2 u_i(x_i(t))$ is negatively defined, whence by virtue of (2.1) and (2.2) for $x_i(t) - \omega_i \neq 0$ we conclude

$$\langle h_i(x_i(t)), \omega_i - x_i(t) \rangle = -\langle \omega_i - x_i(t), \nabla^2 u_i(x_i(t)) (\omega_i - x_i(t)) \rangle > 0.$$  

Thus, denominator in (2.4) is positive and, therefore, the situation of breaking of contracts by the individual $i$ is completely characterized by (2.1) and additional condition

$$\langle h_i(x_i(t)), v_i \rangle < 0.$$

What will take place in the case of several agents, desiring to break off contracts, when a new contract $v$ starts to be realized? In other words, how will the contractual process go, if more than one individual satisfies (2.1)? For all these individuals values $\alpha_i(x(t), v, \Delta t)|_{\Delta t=0} = 1$, but the character of process is determined by their derivatives. It is clear, that the break will happen only if at least one derivative relative to $\Delta t$ is negative, and the measure of breaking is defined by greatest absolute value from negative derivatives. Thus, it is proved the following

**Lemma 2.1** Consider contractual process with partial breaking of barter contracts, satisfying hypotheses (IB), (UB) — Instant Uniform Breaking of all contracts. Let $(x,v)$ be a couple achieved in process at a moment $t \geq 0$, where $x = x(t) = (x_1, \ldots, x_n)$ is allocation and $v = (v_1, \ldots, v_n)$ is a momentary mutually beneficial barter contract, signed among individuals at the moment $t$. A pair $(x,v)$ sets a situation of breaking of contracts if and only if for some $i \in I$ takes place

$$\langle \nabla u_i(x_i(t)), x_i(t) - \omega_i \rangle = 0 \quad \& \quad \langle h_i(x_i(t)), v_i \rangle < 0,$$  

(2.5)

$$h_i(x_i(t)) = \nabla u_i(x_i(t)) + \nabla^2 u_i(x_i(t))(x_i(t) - \omega_i).$$

In such case the local law of contractual process is defined by equation

$$\dot{x}(t) = \lambda(x,v)(x(t) - \omega) + v,$$  

(2.6)

where $\lambda(x,v)$ is minimum of $\lambda_i(x_i,v_i)$, calculated for individuals $i \in I$, satisfying condition $\langle \nabla u_i(x_i(t)), x_i(t) - \omega_i \rangle = 0$; here

$$\lambda_i(x_i,v_i) = \frac{\langle h_i(x_i(t)), v_i \rangle}{\langle h_i(x_i(t)), (\omega_i - x_i(t)) \rangle}.$$  

As it already was noted, if the break of contracts does not occur, the local law of change of a contractual trajectory is set by a rule $\dot{x}(t) = v$. Hence, one can apply the law (2.6) in a general case, if for $\lambda(x,v) > 0$ replace this value by zero. Combining this fact with the result of previous Lemma 2.1, we come to the following definition of proper- contractual trajectories. Let’s define

$$\lambda^{\min}(x,v) = 0 \land \min \left\{ \frac{\langle h_i(x_i(t)), v_i \rangle}{\langle h_i(x_i(t)), (\omega_i - x_i(t)) \rangle} \mid i : \langle \nabla u_i(x_i), x_i - \omega_i \rangle = 0 \right\}.^8$$  

(2.7)

**Definition 2.1** An absolutely continuous map $x(\cdot) : [0, +\infty) \to \mathcal{A}(X)$ is called proper contractual trajectory under hypotheses (IB), (UB), if the following conditions are satisfied:

---

^8Here standardly $a \land b = \min\{a,b\}$. 

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\( \langle \nabla u_i(x_i(t)), x_i(t) - \omega_i \rangle \geq 0, \forall i \in I; \)

(ii) Derivative of the trajectory obeys the law

\[
\dot{x}(t) = \lambda_{\text{min}}(x,v)(x(t) - \omega) + v, \tag{2.8}
\]

where \( v \in L^c \) is mutually beneficial contract, i.e., \( \langle \nabla u_i(x_i(t)), v_i \rangle > 0, \forall i \in \text{supp}(v), \) and value \( \lambda_{\text{min}}(x,v) \) is defined by (2.7).

Notice, that due to this definition proper contractual trajectory is actually described as a solution of some differential inclusion

\[
\dot{x}(t) \in F(x), \ x(0) = \omega
\]
on interval \([0, +\infty)\), where the right hand side obeys (i), (ii).

However for the further analysis of contractual processes this form is not quite convenient. Already for process without breaking of the contracts, for the convergence (to Pareto boundary) one has to require some additional assumptions, ensuring the contractual process is going fast enough. More convenient form is to add in the description of process some trade rule, which unequivocally determines process as a whole.

Let us consider an aggregated proper-contractual trajectory of UBI-process by Definition 2.1. Now we have to specify which of the mutually beneficial contracts is signed by economic agents and now it occurs a map, that it is possible to name a trade rule. Really, the fact that a contract is mutually beneficial depends only on current consumption bundles. This is why one can assume that there is determined a continuous map

\[
v : A(X) \to L^c = \{ v \in L \mid v = (v_i)_{I} : \sum_{i \in I} v_i = 0 \},
\]
such that

\[
\forall i \in I, \ u_i(x_i + v_i(x)) > u_i(x_i) \iff \exists v \in L^c : u_i(x_i + v_i) > u_i(x_i) \ \forall i \in I, \tag{2.9}
\]

and

\[
\exists v \in L^c : u_i(x_i + v_i) > u_i(x_i) \ \forall i \in I \Rightarrow v(x) = 0.
\]
The value \( v(x) \in L^c \) unequivocally specifies the contract, which will be signed at moment \( t \) if \( x(t) = x \), i.e., if allocation achieved by a trajectory at moment \( t \) coincides with \( x \). Notice that when utilities are differentiable instead of (2.9) it is appropriate to apply condition presented for gradients (derivatives by direction):

\[
\forall i \in I \langle \nabla u_i(x_i), v_i(x) \rangle > 0 \iff \exists v \in L^c : u_i(x_i + v_i) > u_i(x_i) \ \forall i \in I.
\]

Really, requirement (2.9) seems quite correct for discrete time presentation, however when time is continuous \( v(x(t)) \) is a momentary contract which defines not only beneficial exchange proportions but also the speed of exchange processes. Integrating \( v(x(t)) \) over time on interval \([t, t + \Delta t]\) one can find a gross contract, obeying (2.9) in the moment \( t \) and when a time of its realization is \( \Delta t > 0 \) (see §1.2).

In closing of this section we would like to do an important remark. The right hand side of equation determining the law of contractual trajectory (2.8) is discontinuous in
general — in spite of it is defined unambiguously and is formulated via continuous functions! Of course this problem (perhaps imperceptible for a first view) is arisen because of parameter $\lambda_{\min}(x, v)$ defined by formula (2.7) vanishes in an open area $x \in A(X)$: $(\nabla u_i(x_i), x_i - \omega_i) > 0$ for all $i$ but in general it is non-zero on its boundary, i.e., at the points $(x_1, x_2, \ldots, x_n) \in A(X)$ where $(\nabla u_i(x_i), x_i - \omega_i) = 0$ at least for one $i$. Thus the law of proper contractual trajectory is described via differential equation with a discontinuous right hand part. Moreover in such a case already the concept of solution requires an accurate definition. Solution is a continuous function of time, which obeys the law of trajectory for almost all time moments. The solution of this kind defines a contractual trajectory in an appropriate way.

Notice that classical theorems on existence, uniqueness and continuous dependence over initial data cannot be applied to the equations with discontinuous right hand parts since their right part do not obey Lipshitz condition. However by now there is a theory of equations of this type in which appropriate theorems (on existence, uniqueness and continuous dependence) have proven and these theorems are applicable to equations describing contractual processes, see [3].

3 Contractual process in $2 \times 2$ economy

Before the description of obtained results we consider one particular example of economy with two individuals, two commodities and Cobb-Douglas utilities. In [22] one can find another similar example where utilities are specified via exponents and there are three equilibria: two of them are locally stable and one is unstable. The examples of this kind are interested because they reveal in Edgeworth box the geometrical course of contractual processes with partial breaking of contracts. One can easily observe that our process is convergent in these cases.

Further an agent is called active at a current time moment $t$ if he/she realizes an breaking of aggregated contract at this moment of time.

Example 3.1 ($2 \times 2$ economy with Cobb-Douglas utilities) Let positive orthant in 2-dimensional plane present individual consumption sets, i.e., $X_i = \mathbb{R}^2_+$, $i = 1, 2$. Let preferences be presented by Cobb-Douglas utilities in logarithmic form, where for convenience $x = (x_1, x_2)$ denotes 1st agent consumption and $y = (y_1, y_2)$ is applied for 2nd:

$$u_1(x_1, x_2) = \frac{1}{4} \ln x_1 + \frac{3}{4} \ln x_2, \quad u_2(y_1, y_2) = \frac{3}{4} \ln y_1 + \frac{1}{4} \ln y_2.$$ 

Consider also the following initial endowments:

$$\omega = (\omega_1, \omega_2) = ((\frac{9}{10}, \frac{1}{10}), (\frac{1}{10}, \frac{9}{10})), \quad \bar{\omega} = \omega_1 + \omega_2 = (1, 1).$$

Then indifference curves for first and second individual going across initial endowments point $\omega_1$ in 1st agent’s coordinate system are described by equations:

$$x_2 = \frac{1}{10} \left( \frac{9}{10x_1} \right)^{\frac{3}{4}}, \quad x_2 = 1 - \frac{9}{10} \left( \frac{1}{10(1 - x_1)} \right)^{\frac{3}{4}}.$$ 

Calculations show that Pareto boundary is a curve determined by equation

$$x_2 = \frac{9x_1}{1 + 8x_1}, \quad 0 \leq x_1 \leq 1.$$
Finally, a maximal surface is composed via two curves and it is the low envelope for them:

\[ x_2 = \left( \frac{3x_1}{40x_1 - 9} \right), \; x_1 > \frac{9}{40} \; \& \; x_2 = \left( \frac{28 - 31x_1}{37 - 40x_1} \right), \; x_1 < \frac{28}{31}. \]

An illustration of this example in Edgeworth box is given in Figure 2.

In considered case proper contractual process is converged to unique equilibrium \((\frac{1}{4}, \frac{3}{4}), (\frac{3}{4}, \frac{1}{4})\). This is developed in the following way: if its trajectory is in limits of maximal surface (shaded area of Figure 2), then individuals are cooperated and signed some barter contracts giving a rise of utilities. A current consumption point is moving this way as long as it starts to leave the maximal surface limits. If a new contract starts to lead the point behind maximal surface and new position is under control of 1st agent (we call him ‘active’ in contractual process), this is the left low part of box restricted by budget line, then this agent partially breaks aggregated contract and a current point of trajectory is projected onto maximal surface along to straight line going at initial endowments point. Analogous thing takes place for second agent if a new contract leads the point to the area under 2nd agent control (right upper part of box behind the budget line). As it is shown in figure in both cases next point of trajectory moves over maximal surface and approaches to equilibrium. Thus in the limit our trajectory achieves equilibrium allocation where 1st agent consumption is \((\frac{1}{4}, \frac{3}{4})\). \[\blacksquare\]

The geometry of contractual trajectories in the considered examples says us that at
least in economy with two individuals and two goods the contractual process has to converge to equilibrium under rather general assumptions.

To prove this hypothesis, let us consider some contractual trajectory \( x(t) = (x_1(t), x_2(t)), t \in [0, +\infty) \) satisfying to Definition 2.1 and defined via some rule of trade

\[
v : \mathcal{A}(X) \to L^c = \{(v_1, v_2) \in (\mathbb{R}^2)^T | v_1 + v_2 = 0\}.
\]

By definition \( v_2(x) = -v_1(x), \forall x \in \mathcal{A}(X) \) and it is enough to set only function \( v_1 : \mathbb{R}^2_+ \to \mathbb{R}^2 \), where the vector \((v_1(x_1), -v_1(x_1)) = v^T(x_1)\) is associated with the instant contract, which is signed the members of coalition \( \{1, 2\} \) at the moment \( t \in [0, +\infty) \) provided that a current allocation is \( x(t) = (x_1, x_2), x_2 = \omega_1 + \omega_2 - x_1 \). In addition, this function should be continuous and to define mutually beneficial contract \( v = v^T(x_1) \): \( v \neq 0 \) if and only if

\[
\exists \nu \in L^c : u_i(x_i + \nu_i) > u_i(x_i), \ i = 1, 2 \quad \& \quad \partial u_i u_i(x_i) > 0, \ i = 1, 2.
\]

Further let us show that for contractual process in economy with two individuals only two following alternatives can be realized.

(i) There is an individual such that since some moment \( \tau \), almost everywhere on \([\tau, +\infty), only he/she can be active (probably both are passive); thus utility of this individual monotonously does not decrease along a trajectory, i.e., for example for the first agent, it has to be

\[
u_1(x_1(t')) \geq u_1(x_1(t)), \ \forall t' \geq t \geq \tau.
\]

(ii) The case described in (i) is not true, i.e., there are monotonously increasing sequences \( t'_{k+1} > t'_k, t''_{k+1} > t''_k, k \in \mathbb{N}, t'_k \to +\infty, t''_k \to +\infty \) when \( k \to +\infty \), such that at the moments \( t'_k \) the 1st individual is active, and \( t''_k \) are the moments of 2nd agent activity, \( k = 1, 2, \ldots \). Thus, utilities of both individuals can oscillate, growing and decreasing, and this situation does not change when time elapses.

Analysis of alternatives (i), (ii) is realized in two subsequent lemmas. The first establishes that if alternative (i) is true then every limit point of a contractual trajectory is Pareto optimal. The second alternative causes the main difficulties and second lemma states that in some sense there is the monotonicity of utilities along a trajectory but it has “piecewise” character.

**Lemma 3.1** Let alternative (i) be fulfilled. Then each limit point of a contractual trajectory is Pareto optimal. Hence, every interior limit point is equilibrium one.

The proof of Lemma 3.1 is placed in Appendix.

**Remark 3.1** One can easy to see from the proof that this lemma is also true for economy with any number of agents and commodities. Moreover, if economy has only two agents and alternative (i) is true then all limit points have equal utilities for both agents, that for strictly concave functions is possible only if allocations are equal. Applying the argument of Lemma 3.1 proof one can also conclude the convergence of contractual process without break of contracts. However now it will be convergence to some Pareto optimal allocation.
Lemma 3.2 Let alternative (ii) be fulfilled. Then there are exist two monotonously increasing sequences of the moments of time $\tau_k^i$, for $i = 1, 2$, such that $\tau_1^i < \tau_2^i < \tau_1^{i+1}$, $\langle \nabla u_i(x_i(\tau_k^i)), x_i(\tau_k^i) - \omega_i \rangle = 0$, $i = 1, 2$, $\forall k \in \mathbb{N}$ and

$$u_i(x_i(\tau_k^i)) < u_i(x_i(\tau)) < u_i(x_i(\tau_k^{i+1})), \quad \forall \tau \in (\tau_k^i, \tau_k^{i+1}), \quad i = 1, 2$$  \hspace{1cm} (3.1)

holds.

The proof of Lemma 3.2 is placed in Appendix. The contents of this lemma, possible and impossible dynamics of proper-contractual trajectories are demonstrated in Figure 3. Here there are presented two points $x_1(\tau_1^1)$, $x_1(\tau''')$ of a trajectory in which in different but closest time moments $\tau_1^1$ and $\tau'''$ different agents are active. It is shown, that the only fragment of trajectory, represented in the top part of figure is possible — all other variants result to an impossible cycle contradicting with the choice of the moments $\tau_k^i$.

The main result of the section is presented in the following theorem, that states the convergence of contractual trajectories to equilibrium in economy with two agents and two goods.

Theorem 3.1 Let economy have 2 agents, 2 commodities and (S) be satisfied. Then for any continuous trading rule the contractual trajectory by Definition 2.1 converges to some proper-contractual allocation. Hence, in conditions when equilibrium allocations coincide with proper-contractual ones, every proper-contractual trajectory converges to equilibrium.
4 Convergence of benevolent UB-processes

In this section we consider a special class of contractual UBI–processes which has a convergence property: it will be shown that under specific assumptions these processes generically converge to equilibrium.

**Definition 4.1** A rule of trade \( v : \mathcal{A}(X) \to L^c \) is called benevolent, if \( v(x) \) does not attract the break of contracts in all situations when a mutually beneficial exchange without break is possible.

Contractual UB–process is called benevolent, if it is defined by a benevolent rule of trade.

In substantial terms Definition 4.1 means that before the individuals sign a new contract, they carefully investigate opportunities for a mutually beneficial exchange being aimed to find a contract without subsequent break of made earlier contracts. The signing of a contract with subsequent break is carried out only if there is no any other opportunity to get an agreement.

Formally, for process by Definition 2.1 the concept of a benevolent rule of trade requires performance of the following conditions.

Let \( x \in \mathcal{A}(X) \) be some allocation stable relative to the break of aggregated contract \( x - \omega \) and let

\[
\mathcal{T}^a(x) = \{ i \in \mathcal{I} \mid \langle \nabla u_i(x_i), x_i - \omega_i \rangle = 0 \} \neq \emptyset
\]

be nonempty set of all active individuals. Let us define

\[
W^{fr}(x) = \{ w \in L^c \mid \langle \nabla u_i(x_i), w_i \rangle > 0, \forall i \in \mathcal{I} \& \langle h_i, w_i \rangle > 0, \forall i \in \mathcal{T}^a(x) \}. \tag{4.1}
\]

This is the set (possible empty) of all mutually beneficial contracts that being signed do not attract the break of aggregated contract \( x - \omega \).\(^9\) If \( W^{fr}(x) \neq \emptyset \) then

\[
\langle h_i, v_i(x) \rangle \geq 0, \forall i \in \mathcal{T}^a(x) \& \langle \nabla u_i(x_i), v_i(x) \rangle > 0, \forall i \in \mathcal{I}. \tag{4.2}
\]

Notice that all allocations, where mutually beneficial exchange without break is impossible, form a closed set in \( \mathcal{A}(X) \). For the checking this fact it is enough to see that allocations where exchange without break is possible form an open subset in \( \mathcal{A}(X) \). Really if \( x \in \mathcal{A}(X) \) has the last property then by continuity of functions one can find a neighborhood \( V_x \) for \( x \) such that \( \mathcal{T}^a(y) \subseteq \mathcal{T}^a(x), \forall y \in V_x \). Now if \( w \in W^{fr}(x) \neq \emptyset \) then there is a neighborhood \( y \) (possibly less then \( V_x \)) such that for all \( y \) from the neighborhood \( w \in W^{fr}(y) \Rightarrow W^{fr}(y) \neq \emptyset \).

We begin the analysis from detailed research of a general case of benevolent trajectories. With this in mind we first study situations in which a mutually beneficial exchange without break of contracts is impossible. The following lemma describes necessary conditions that such situation takes place for some allocation.

**Lemma 4.1** Let \( x \in X \cap \text{int}\mathcal{A}(X) \) be an allocation stable relative to the partial break of gross contract \( x - \omega \). Let standard assumptions be satisfied and let mutually beneficial
exchange without the subsequent break of contracts is impossible, i.e. \( W^F(x) = \emptyset \). Then there exists a vector \( p \neq 0 \) so that for each individual \( i \in I \) there are numbers \( \alpha_i \geq 0, \beta_i \geq 0 \) such that
\[
p = \alpha_i \nabla u_i(x_i) + \beta_i h_i(x_i), \quad \forall i \in I
\]
(4.3)
and complementarity slackness conditions
\[
\beta_i \cdot \langle \nabla u_i(x_i), x_i - \omega_i \rangle = 0, \quad \forall i \in I
\]
are fulfilled.

Complementarity slackness conditions just serve a convenient form to describe the fact that for each passive individual (if \( \langle \nabla u_i(x_i), x_i - \omega_i \rangle > 0 \)) the value \( \beta_i \geq 0 \) should be zero while for an active agent (if \( \langle \nabla u_i(x_i), x_i - \omega_i \rangle = 0 \)) it can be strictly more than zero. Further, it is easy to see that up to normalization of \( p \) the values \( \alpha_i \) and \( \beta_i \) in the formula (4.3) are unequivocally defined since \( \nabla u_i(x_i) \) and \( h_i(x_i) \) are the non-collinear couple of vectors for any active individual in the current allocation. It allows us among the active individuals correctly to identify the agents who can really influence a course of contractual process via a break of the contracts. The proof of this and subsequent Lemma 4.2 is placed in Appendix.

So, we shall call an active individual \( i \) really active, if \( \beta_i > 0 \). Accordingly, if \( \beta_i = 0 \) for an active agent then he/she may be called as fictitiously active (locally in process behavior of these individuals similar to passive ones). Also let’s name the individual really passive, if he/she is passive or fictitiously active.

The property of a trajectory to be benevolent is quite qualified requirement for a rule of trade. In particular, it is easy to see that the vector \( p \) which exists by Lemma 4.1 being normalized as \( \|p\| = 1 \) is a continuous function of the current point of the trajectory.

**Lemma 4.2** Let \( x(t), \ t \geq 0 \) be a benevolent UB-trajectory by Definition 2.1 and let standard assumptions be satisfied. Then the vector \( p = p(x(t)), \|p\| = 1 \), existing by Lemma 4.1 at all points of trajectory \( x(t) \) where beneficial exchange without contracts breaking is impossible is continuous function in its domain.

It is clear, that in lemma conditions when the continuous dependence of \( p \) from the current point of the trajectory is specified, the same property is fulfilled for the coefficients \( \alpha_i, \beta_i \) of decomposition (4.3): they are also continuously depend on time. It follows from the fact (already mentioned) that vectors \( \nabla u_i, h_i \) being calculated at required points are non-collinear.

Further we will be interested in some specific properties of proper-contractual trajectories by Definition 2.1 (not necessarily benevolent!). In fact, we need to clear those situations, when more than one active individual may exist at the points of a trajectory. With this in mind we recall that at the current point of a trajectory \( x(t) \) the measure of break of the gross contract is defined as a minimum (provided that it is less than zero, otherwise a break does not occurs) of some values determining desirable break for the active individuals. Desirable break of gross contract for the agent \( i \) is defined by value
\[
\lambda_i(x(t), v(x(t))) = \frac{\langle h_i(x_i(t)), v_i(x_i(t)) \rangle}{\langle h_i(x_i(t)), \omega_i - x_i(t) \rangle},
\]
see Lemma 2.1. However for two individuals simultaneously define the size of break of the contracts in *nearest subsequent* after \( t \) moments of time, it is necessary that the measure of desirable break of gross contract coincides with a general minimum and, therefore, both measures should coincide among themselves. This motivates the following definition.

**Definition 4.2** A contractual trajectory \( x : [0, +\infty) \to \mathcal{A}(X) \) (process) is called **non-degenerate** if for all non-equilibrium points \( x(t) \) if \( \langle \nabla u_i(x(t)), x_i(t) - \omega_i \rangle = \langle \nabla u_j(x_j(t)), x_j(t) - \omega_j \rangle = 0 \) for some \( i \neq j \), then

\[
\lambda_i(x(t), v(x(t))) = \frac{\langle h_i(x_i(t)), v_i(x_i(t)) \rangle}{\langle h_i(x_i(t)), \omega_i - x_i(t) \rangle} \neq \frac{\langle h_j(x_j(t)), v_j(x_j(t)) \rangle}{\langle h_j(x_j(t)), \omega_j - x_j(t) \rangle} = \lambda_j(x(t), v(x(t))).
\]

A trading rule is called **non-degenerate** relative to initial endowments allocation \( \omega = (\omega_1, \ldots, \omega_n) \) if a generated trajectory with initial data \( x(0) = \omega \) is non-degenerate.

Non-degenerate contractual trajectories are easier in the analysis, because in each moment of time only one individual sets a measure of break of gross contract, i.e. he/she can be considered to be as an active “leader” of contractual process. However, when time elapses for non-degenerate processes of a general form the leader can be changed. Below we will see that it does not occur for the benevolent trajectories.

With this in mind we first show, that for any not stabilized trajectory the set of those moments where there are two or more active agents has a structure similar to discrete one\(^{10}\).

**Lemma 4.3** Let \( x(t) \) be a non-degenerate UB-contractual trajectory by Definition 2.1 and 4.2. Let in the moment \( \tau \) the derivative \( \dot{x}(\tau) \neq 0 \) and let an agent \( i \) is active, i.e. \( \langle \nabla u_i(x_i(\tau)), x_i(\tau) - \omega_i \rangle = 0. \) Then for some \( \varepsilon > 0 \) and for all points from \( (\tau, \tau + \varepsilon) \) only one of the following alternatives takes place:

(i) If \( \lambda_i(x(\tau), v(x(\tau))) = \lambda^{\min}(x(\tau), v(x(\tau))) < 0 \) only individual \( i \) is active and all other agents are passive.

(ii) If \( \lambda_i(x(\tau), v(x(\tau))) = \lambda^{\min}(x(\tau), v(x(\tau))) = 0 \) individual \( i \) may be active but it is not certainly the case however all other agents are certainly passive.

(iii) If \( \lambda_i(x(\tau), v(x(\tau))) > \lambda^{\min}(x(\tau), v(x(\tau))) = 0 \) for each active individual then in interval \( (\tau, \tau + \varepsilon) \) all agents are passive.

One can find the proof of Lemma 4.3 in Appendix. Notice that the continuity of trade rule is quite important this lemma to be true.

Further we show that while equilibrium is not attained along a benevolent trajectory there is only one individual that can be “really” active. Remember that we have named really active such active individual which has parameter \( \beta_i \neq 0 \) in the relation (4.3) from Lemma 4.1.

**Theorem 4.1** Let \( x(t) \) be non-degenerate benevolent trajectory and let standard assumptions be fulfilled. Then only one of the following alternatives can be true:

\(^{10}\)Strictly speaking, this set can have limit points, however it does not influence the subsequent analysis.
(i) There are no (almost) time moments on the interval \([0, +\infty)\) when the breaking of gross contract \(x(t) - \omega\) is realized, i.e., for almost all time moments during contractual process all individuals are passive.

(ii) There exists such time moment \(\tau > 0\) that for almost all time moments on the interval \([0, \tau)\) all individuals are passive and the contrary at the moment \(\tau\): all individuals are active, i.e., \(x(\tau)\) is an equilibrium.

(iii) There exist time moments \(\tau_1 > 0\) and \(\tau_2 > \tau_1\) such that for almost all time moments on the interval \([0, \tau_1)\) all individuals are passive and there is the only real active individual on the interval \([\tau_1, \tau_2)\) and if \(\tau_2 \neq +\infty\) then at the moment \(\tau_2\) all individuals are active, i.e., \(x(\tau_2)\) is an equilibrium.

Theorem 4.1 describes rather important properties of non-degenerate benevolent trajectories which allow us to conclude the convergence of this type trajectories to an equilibrium.

Corollary 4.1 Let the standard assumptions be fulfilled. Then any benevolent rule of trade generating non-degenerate contractual process defines proper-contractual UB-trajectory, for which all limit points are equilibria. Thus non-degenerate benevolent processes are quasi-globally stable.

Proof of Corollary 4.1. If alternative (ii) or (iii) when \(\tau_2 < +\infty\) are realized then the convergence of a contractual trajectory to equilibrium is obvious. If the alternative (i) or (iii) with \(\tau_2 = +\infty\) are realized then we are in the condition of alternative (i) from previous section (see page 20) and now we can apply Lemma 3.1 and Remark 3.1 that proves equilibrium properties of any limit point.

One of main results is presented in the following theorem on convergence of non-degenerate benevolent UB-processes. Recall that an economy is said to be regular if the matrix of the differential of excess demand function has maximal rank (equal to \(l - 1\) where \(l\) is number of commodities) for every equilibrium price vector. It is known that every regular economy has a finite number of equilibria and that almost all economies of \(C^2\) class are regular, see e.g. [9].

Theorem 4.2 Let \(E\) be a regular economy and the standard assumptions are fulfilled. Then any non-degenerate benevolent UB-processes converges to an equilibrium.

As a corollary of this theorem applying Thom’s theorems on density and openness of transversal sections it seems possible to prove the following result on generic convergence of benevolent contractual processes to an equilibrium.

Corollary 4.2 For almost all economies of \(C^2\)-class every benevolent contractual UB-process converges to an equilibrium.
Conclusion

The paper presents the results of the following investigations: Contractual processes are in detail described and, first of all, proper-contractual ones — they are processes, in which partial breaking of the contracts is allowed. With this in mind several basic hypotheses determining the character of contracts’ breaking process are formulated in a general form and for major particular cases. They are the following:

- (IB) — instantaneous breaking of the contracts;
- (UB) — uniform breaking of all contracts;
- (CUB) — uniform breaking of gross within-coalitional contracts.

Combinations of these hypotheses result to the proper-contractual trajectories of different kinds of generality. Under (IB) and (UB) contractual trajectory turns out aggregated, under (IB) and (CUB) — coalitional-contractual; there are given formal and mathematically reasonable definitions. In my opinion, the coalitional-contractual trajectory should serve the central concept in further researches. At last, concept of trade rule is introduced; this is a map, unequivocally determining mutually beneficial contract for the current consumption plans, having some additional good mathematical properties. By use of a trade rule a contractual trajectory of each mentioned kinds is unequivocally determined. The special type of benevolent rules of trade is stood out as rules which determine a new contract allowing the break of gross barter contract only if being realized every new mutually beneficial contract involves contracts’ breaking. Just for this class of benevolent processes the basic positive results about convergence were received.

Presented analysis of convergence of contractual trajectories has given the following results:

a) For the economies with 2 individuals and 2 commodities convergence of proper contractual processes relative to any continuous trading rule has proven under rather general assumptions.

b) The theorem on convergence to equilibrium of non-degenerate benevolent UB-contractual processes has been proven.

c) A series of model examples are presented demonstrating specific properties of contractual processes in different cases. The most interesting example presents economy with 4 agents and 2 commodities where coalitional contractual process (CUB) is cycling.

5 Appendix

5.1 Survey of the literature on equilibrium stability

There is a vast economic literature, devoted to the research of processes driving a multiproduct economy to competitive equilibrium. By now one can mark out at least five approaches to explain market dynamics, they have own comparative advantages and shortcomings. These approaches are:
(i) Tâtonnement processes of equilibrium prices of Walrasian\textsuperscript{11} type. This is tâtonnement, where a current disequilibrium prices change by the law of excess demand: if it is positive the price increases if negative then price decreases.

(ii) Processes, in which the law of change of prices is defined due to Jacobi matrix (differential) of excess demand function. The first process of this type was suggested by Smale [33].

(iii) Disequilibrium models of trade processes among consumers; among them Hahn–Negishi process [10] and Edgeworth processes by Uzawa [34].

(iv) Edgeworth processes. They are the processes of commodity exchange without prices, they are based on a mutually beneficial barter (irrevocable) among the members of any coalition of consumers. As time elapsed the coalitions of agents participated in exchange may vary and run some class of permitted coalitions (some coalitions can be forbidden, may be because of that the formation of them is incredible from the essential point of view and exchanges are not realized).

(v) Strategic approach, where equilibrium and competition are examined from purely game theoretical point of view.

Below we consider the specified approaches in more details in (limited) frameworks of well-known Arrow-Debreu type economy of pure exchange. For convenience of an exposition we begin with introduction of notations and reminder of concepts and notions.

Let us consider a typical exchange economy described in section 1.1. represented as a triplet:

\[ \mathcal{E} = (\mathcal{I}, E, (X_i, u_i(\cdot), \omega_i)_{i \in \mathcal{I}}). \]

Traditionally the concept of competitive equilibrium is usually expressed in the terms of equality between demand and supply or simply as the equality to zero of excess demand. Individual demand \( d_i(p) \) of agent \( i \) for prices \( p = (p_1, \ldots, p_l) \neq 0 \) is the solution of utility maximization problem

\[ u_i(y) \rightarrow \max, \quad \text{subject to } py \leq p\omega_i, \quad y \in X_i. \]

If \( p \gg 0 \) this problem always has a solution and therefore the map \( d^i(\cdot) \) is well defined on \( \mathbb{R}^l_{++} \). Together with individual demand it is sometimes convenient to consider individual excess demand function, determined by formula \( z^i(p) = d^i(p) - \omega_i \).

In a context of assumptions for Arrow-Debreu model the map (function) of excess demand for a given prices is defined as a sum of individual solutions of consumer problems for all agents of economy (total demand) minus the total supply. In such a way the vector \( D(p) = \sum_i d^i(p) \) is called total or aggregated demand. As soon as in a context of exchange model supply is fixed and equal to \( \bar{\omega} = \sum_i \omega_i \), then excess demand under the prices \( p \) is

\[ Z(p) = D(p) - \bar{\omega} = \sum_i d^i(p) - \sum_i \omega_i = \sum_i z^i(p). \]

\textsuperscript{11}In this process market prices on different goods are changed simultaneously, this is a modification of original Walrasian idea suggested by Samuelson.
Obviously, the maps of demand and excess demand are correctly defined on area of change of the prices $\mathbb{R}_{++}^l$. Moreover, a vector $p \gg 0$ is the prices of equilibrium if and only if $Z(p) = 0$ (or, equivalently, $Z(p) \leq 0$).\footnote{In general, the excess demand map can be a point-to-set mapping however for simplicity we shall assume below that demand is single-valued (it is provided by strictly quasi-concave utilities).}

Besides under the natural model assumptions (classical convex, continuous preferences) function $Z(\cdot)$ is continuous on its domain $\mathbb{R}_{++}^l$. Moreover, where it is necessary, we shall assume without special mentioning that it is differentiable in a proper degree. By construction, the function of excess demand $Z(p)$ is homogeneous of degree $0$ and satisfies to Walras law:\footnote{For Arrow-Debreu model it is provided due to local non-satiated preferences.}: $\langle p, Z(p) \rangle = 0, \forall p \gg 0$. For the existence of equilibrium and also to provide the convergence of a number of processes of price changes to equilibrium, in addition it is usually assumed boundary conditions. The basic example of this kind condition is the following: if $p^m \to p^0 \in \partial \mathbb{R}_{++}^l$, $p^0 \neq 0$ for $m \to \infty$, then $Z_j(p^m) \to +\infty$ for $p^0_j = 0, j = 1, \ldots, l$. In the limits of this section we always shall think that it is true everywhere, where it is necessary.

Further we proceed to the direct description of specified above processes and results.

### 5.1.1 Walrasian tâtonnement process of equilibrium prices

There is the vast literature, devoted to the study of this type processes, now we note only two reviews [9], [29], where one can find the detailed description of problems and results. Further first of all we describe process formally. The economy is described in most aggregated form via to excess demand function $Z : p \to Z(p)$, defined for all positive prices $p \gg 0, p \in \mathbb{R}^l$.

In a general case ‘tâtonnement’ is described as a process of prices $p(t) = (p_j(t))_{j=1,\ldots,l}$, $t \geq 0$ changes, as the solution of the following system of the differential equations:

$$
\dot{p}_j(t) = F^j(Z_j(p)), \quad j = 1, \ldots, l.
$$

(5.1)

Here it is always assumed that all functions $F^j(\cdot)$ are sign-preserving, i.e., we have $Z_j(p)F^j(Z_j(p)) > 0 \iff Z_j(p) \neq 0$. Exactly this property of right hand side of (5.1) implies that the price of $j$-th commodity adjust in the same direction as excess demand for that commodity: it increases when excess demand is positive and vice versa. It is commonly assumed in addition [9] that $\frac{d}{dx}F^j(x) > 0$ for points from an appropriate area for process (5.1). It is known [9] that the solution of our system exists and is unique for every initial data $p(0) \gg 0$. In classical tâtonnement it is presumed a simplest form of functions $F^j(Z_j(p)) = Z_j(p)$, i.e., in classic approach $F^j(\cdot)$ is identity map of real numbers into itself. In this case the system of differential equations (5.1) turns into $\dot{p} = Z(p)$.

There are also considered some “intermediate” variants where, for example, it is supposed $F^j(Z_j(p)) = k_jZ_j(p)$ for some real $k_j > 0, j = 1, \ldots, l$.

Essential treatment of tâtonnement process (5.1) is commonly based on a hypothesis of fictitious auctioneer, which, likely as it occurs in real auctions, raises the price when demand exceeds supply and, accordingly, reduces price, if the demand is less than supply. An auctioneer is not revealed in model (Arrow-Debreu type) economic agent, this is some impersonal being, whose actions reflects an invisible hand of the market. It is not quite clear in fact how the markets really work in a disequilibrium situation, during to
search of equilibrium, because there are no presented revealed microeconomic models of process and of agents’ behavior out of equilibrium. In the literature a hypothesis about the existence of auctioneer and similar constructions are criticized and are recognized as unrealistic, for example, see [16] p. 195–198, [4], p. 19–26. Fisher [4] have noted the following difficulties in the process (5.1) interpretation: First, “... It has nothing to do with the question of whether or not trade, consumption, or production takes place out of equilibrium” (actually, exchange is possible only when equilibrium is attained!). Second, “...we know very little about how individuals do or ought to behave when equilibrium is not presented; hence, the resort to an aggregate equation”. Finally, “... in the unrealistic world of no trading out of equilibrium ... individuals take action to make their excess demand effective. ... they can take such action which ... implies that they have something of value which they can and do sell so as to have something to offer when they buy...” So, out of equilibrium individuals have to buy and sell to reveal their excess demand, for the price changes to be going, but actually they can do it only under equilibrium prices.

Summarizing, one can conclude that equilibrium theory needs an adequate dynamic theory, in which framework the process of equilibrium prices searching has to be revealed. The subject of criticism is also rather rigorous conditions, under which one can guarantee the convergence of process (5.1) to equilibrium. Results on convergence and stability of process (5.1) mainly are based on the property of gross substitutability of excess demand functions, in various forms of generality, or just on the axiom of revealed preferences and others. Let us consider these assumptions in more details, see [23].

• A function\(^{14}\) \(Z(\cdot)\) has the property of gross substitutability (GS–property), if for any prices \(p’\) and \(p\) such that \(p’_m > p_m\) is true for some \(m\) and \(p’_k = p_k\) for \(k \neq m\), then \(Z_k(p’) > Z_k(p)\) takes place for all \(k \neq m\), \(k = 1, \ldots, l\).

For a differentiable function \(Z(\cdot)\) the condition of gross substitutability takes the form \(\partial Z_k(p)/\partial p_m > 0, \forall p \gg 0, \forall k \neq m\).

• An excess demand function \(Z(\cdot)\) satisfies to weak axiom (WARP) of revealed preference, if for any couple of vector-prices, \(p\) and \(p’\),

\[
Z(p) \neq Z(p’) \ & \ pZ(p’) \leq 0 \Rightarrow p’Z(p) > 0
\]

takes place.

Being applied to (aggregated) excess demand function, the properties of gross substitutability and (weak) revealed preference are, in general, non-equivalent and rather strong requirements. However both of them have common important corollaries:

(i) The set of equilibrium prices is convex.

(ii) If \(p^*\) is equilibrium price, then \(p^*Z(p) > 0\) for all \(p \gg 0\) which are not proportional to \(p^*\).

In particular, property (ii) allows easily to understand why process (5.1) is converged, if one of the specified conditions is carried out. Really, in the simplest case of (5.1), when \(\dot{p} = Z(p)\), it is enough to differentiate by \(t\) the function of squared Euclidean distance between a current prices and equilibrium prices, \(||p(t) - p^*||^2 = \sum_{j=1}^l (p_j(t) - p^*_j)^2\). By virtue of Walras law, we obviously have \(\frac{d}{dt}||p(t) - p^*||^2 = 2(p(t) - p^*)\dot{p} = -2p^*Z(p) < 0\).

\(^{14}\)In a context of an exchange model the gross substitutability of excess demand function and demand function are equivalent.
So, we see that the distance between a current vector of prices and equilibrium ones decreases when time is going.

Finally we would like to mention one more classical condition, providing the local convergence of tatonnement process, this is the property of diagonal domination of Jacobi matrix $D_pZ^-(p^*) = A$, without the last row and column, of excess demand function $Z(p)$ in a point of equilibrium $p^*$. Formally, diagonal domination means

$$\exists h = (h_1, \ldots, h_{l-1}) \geq 0 : \forall j \ h_ja_{jj} < \sum_{k \neq j} h_k|a_{jk}|.$$ 

Gross substitutability implies this property, but the opposite is false. Moreover, in literature there are not known other examples of diagonal domination. The diagonal domination and other similar requirements, e.g., see Theorem 1.7 in [9], implies that eigenvalues of $A$ have negative real parts and this provides a local stability of price adjustment process.

5.1.2 Processes of the prices change, using Jacobi matrix of excess demand function

First of all we would like to notice, that a large part of the critical remarks, made relative to Walrasian processes, can be also addressed to the processes of this type.

Smale [32] investigated the convergence of prices changes process, based on (global) Newton method, which is usually applied to find a solution of system of the nonlinear equations. The process is determined as:

$$\dot{p} = -\lambda(p) [D_pZ^-(p)]^{-1}Z^-(p). \tag{5.2}$$

Here $Z^-(p)$ is excess demand for all goods excepting (for example) the last one, and $D_pZ^-(p)$ is Jacobi matrix of excess demand function, excepting the last row and column. It is supposed, that the sign of functions $\lambda(p)$, entering as a factor in the right part of (5.2), coincides with the sign of $(-1)^{l-1}\det[D_pZ^-(p)]$. If $D_pZ^-(p)$ is non-singular matrix in the domain of $p$ changes, then process (5.2) can be rewritten in an explicit form

$$\dot{p} = -\lambda(p) [D_pZ^-(p)]^{-1}Z^-(p).$$

Smale [32] proved, that this process converges to equilibrium for any aggregated excess demand function, if the initial prices $p(0) \neq 0$ are on the boundary of $\mathbb{R}^l_+$ (area of prices change), except for a set of zero measure (with the account of normalization), and under additional requirement that $D_pZ^-(p)$ is non-singular in effective area of prices change (these are positive and normalized by the last component). Certainly, this is remarkable result, however its weak side is too large informational requirements. Really, at each time moment the process of prices change requires a knowledge not only excess demand, but

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15For the first time the method was offered and partially investigated by Arrow and Hahn [1].

16To the last row and column there corresponds a commodity, which is used as a numeraire good. The elimination of a row and column is necessary, to allocate square nonsingular submatrix in $J[Z(p)]$; since excess demand is homogeneous, $J[Z(p)]p = 0$ and, therefore, matrix $J([p])$ is always singular.

17Certainly, it is necessary also to postulate other properties of $\lambda(p)$, ensuring existence and uniqueness of solution (5.2).

18Here the economy completely is set by function $Z(p)$. 
also Jacobi matrix, i.e., the change of price in the market explicitly depends on how the prices on other markets are changed.

Kamiya [13], developed Smale’s approach, and has offered process, defined as

\[
\begin{bmatrix}
\frac{D_p Z^- (p)}{||D_p Z^- (p)||} - \frac{I}{||p - p(0)||}
\end{bmatrix} \dot{p} = -\lambda(p) Z^- (p).
\]  

Here, as well as in Smale’s process, \( D_p Z^- (p) \) is Jacobi matrix of excess demand without the last column and row, \( p(t) \) is \((l - 1)\)-dimensional vector-function of the prices without last component \( p_l (t) \), such that \( ||p(t)|| \leq 1 \) and \( p(0) \) is an initial vector of the prices of the same dimension. It is assumed the sign of real-valued function \( \lambda(p) \), placed in the right part of (5.3), coincides with the sign of \( \det \left[ \frac{I}{||p - p(0)||} - \frac{D_p Z^- (p)}{||D_p Z^- (p)||} \right] \). Using methods, suggested by Smale, Kamiya [13] proves, that the process (5.3) converges to equilibrium for almost all initial data \( p(0) \) from the interior of \( \mathbb{R}^{l-1}_+ \).

Mukherji [24] investigated another process, using Jacobi matrix of aggregated excess demand function:

\[
\dot{p} = -J[Z(p)]^\top Z(p).
\]

He has shown in [24], that the process (5.4) belongs to a group of so-called locally effective processes (LEPM): the processes of this type converge to any (regular\footnote{Jacobi matrix of excess demand at equilibrium point has the maximal rank equal to \( l - 1 \).}) equilibrium locally (i.e. for the equilibrium prices there exists a neighborhood, such that if \( p(0) \) in the neighborhood, the process converges to the equilibrium).

Concerning all described above processes, and also other processes from this group,\footnote{For example, for orthogonal Newton process, described in [12].} it is possible to state one common remark: all of them require too much information. Moreover, Saari and Simon [30] have proved, that this is unavoidable property of any LEPM-process, i.e., actually, it is necessary condition for the process to be locally effective for (almost) any function of aggregated excess demand. In relationship we would like to recall Sonnenschein-Debreu-Mantel results, see survey [31], about representation of a general aggregated excess demand function as an excess demand function for Arrow-Debreu model. They show, that any continuous, homogeneous and obeying Walras law function allows representation in the specified for Arrow-Debreu model form, the model where number of the agents is equal to the number of commodities. In so doing the utilities of individuals may be classical: continuous, strictly concave, monotonous and, moreover, homogeneous (degree 1). Thus, one can go to the following conclusion: any locally effective mechanism of prices change based on excess demand function for Arrow-Debreu model is informational requiring and, with necessity, has to use (whole!) Jacobi matrix \( J[Z(p)] \) of excess demand. In particular, making comments to Smale’s process, Hahn ([9], p. 767) replies: “Obviously these results are interesting as algorithms and not as models of invisible hand.”

### 5.1.3 Disequilibrium models of trade processes

There are at least two disequilibrium processes known in literature in a context of a pure exchange model, see [4], [25]. These are Edgeworth process by Uzawa [34], and Hahn’s process [10] (so-named by Negishi in [27]). It is common feature of both processes, that
endowments are varied at the time, i.e., initial endowments \( \omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{R}_+^n \) is the function of time, \( \omega : [0, +\infty) \to \mathbb{R}_+^n \) (\( n \) is a number of agents). As well as in Walrasian processes (tâtonnement), real consumption comes only at the end of the process, where it is described by a limiting point of \( \omega(\cdot) \). Further we consider other specific features of processes.

Common properties. The prices change according to excess demand:

\[
p_j(t) = \begin{cases} 
F^j(Z_j(p, \omega(t))), & \text{unless } p_j = 0 \& Z_j(p) < 0; \\
0, & \text{if } p_j = 0 \& Z_j(p) < 0.
\end{cases} \tag{5.5}
\]

Here the functions \( F^j(\cdot) \) satisfy to the usual requirements: continuity and sign-preservation.

There is postulated a law of change of the initial endowments, which can be also treated as (current) allocation of consumed resources \( \omega : [0, +\infty) \to \mathbb{R}_+^n \), and this map has to obey the requirements:

\[
\forall i \in I, \quad \dot{\omega}_i(t) = g_i(p(t), \omega(t)) - \omega_i(t), \quad \sum_{i=1}^n \omega_i(t) = \sum_{i=1}^n \omega_i(0), \quad \forall t \in [0, +\infty), \tag{5.6}
\]

where all functions \( g_i(p(t), \omega(t)) \) are assumed to be continuous and, in addition, to satisfy to “No Swindling” condition:

\[
\forall i \in I, \quad p(t)\dot{\omega}_i(t) = 0 \iff p(t)g_i(p(t), \omega(t)) = p(t)\omega_i(t), \quad \forall t \geq 0. \tag{5.7}
\]

Essentially, in both processes the functions \( g_i(p(t), \omega(t)) \), \( i \in I \) set a rule of trade (trading or transaction rule). Other requirements in processes differ.

Hahn’s process. A specific requirement is the assumption that markets are orderly:

\[
\forall t \geq 0 \quad z_j^i(p(t), \omega(t))Z_j(p(t), \omega(t)) > 0, \quad j = 1, 2, \ldots, l, \tag{5.8}
\]

unless the case \( z_j^i(p(t), \omega(t)) = 0 \) for all \( i = 1, 2, \ldots, n \). Here \( z_j^i(p(t), \omega(t)) \) is individual excess demand of \( i \)'s agent for the commodity \( j \) under current prices and endowments \( \omega_i(t) \). This requirement means, that if the market of a product \( j \) is not balanced, then all agents have positive excess demand or supply (there are hence only unsatisfied demanders or suppliers for any given good).

Edgeworth’s process by Uzawa. It is supposed, that endowments (here it is a current consumption) are changing so that monotonous growth of utility of each individual goes, at least for one in strictly form, if it is possible in general; everything under constrains (5.6), (5.7). Formally it is defined as: \( \forall t \geq 0 \)

\[
u_i[g_i(p(t), \omega(t))] \geq u_i[\omega_i(t)], \quad \forall i, \tag{5.9}
\]

\[
u_i[g_i(p(t), \omega(t))] = u_i[\omega_i(t)], \quad \forall i \iff g_i(p(t), \omega(t)) = \omega_i(t), \quad \forall i & \& 
\]

\[
\forall (x_1, \ldots, x_n) \in \mathbb{R}_+^n : p(t)x_i = p(t)\omega_i(t) \forall i & \& \sum x_i = \sum \omega_i(t), \\
\exists k : u_k(x_k) > u_k(\omega_k(t)) \Rightarrow \exists i : u_i(x_i) < u_i(\omega_i(t)).
\]

Thus, at each current moment of time, a state of economy changes if and only if it appears possible mutually beneficial exchange within the framework of budget constrains.\(^\text{21}\)
Both described processes (Hahn’s process and Edgeworth’s process by Uzawa) are converged to some Pareto optimal allocation under more or less standard assumptions, including boundary condition (provides a movement of a trajectory in $\mathbb{R}^{n_{++}}$), and additional assumption about strict concavity of utility functions, see [10], [34]. In so doing price processes are also converged (for Edgeworth’s process by Uzawa see [26], [25]) and the limiting prices are the prices of equilibrium for the given limiting resources allocation (here this is initial and final allocation simultaneously), this is a situation of no-trade.

The detailed description of specified disequilibrium processes one can find in [1] (part 13), [9], [4], [25], where their criticism is also contained. For example, Fisher [4], being an advocate of Hahn’s process, criticized Edgeworth’s process by Uzawa in the following way. First, it is not clear, why Pareto improving trade actually will takes place whenever such a situation arises. The reason is that it is possible that all coalitions of the agents, capable to do such mutually beneficial exchange, can have too large size, and small coalitions (bilateral or trilateral or quadrilateral trade and so on) are unable to carry out Pareto-improving exchange. Admitting an opportunity of exchange in the huge coalitions, we impose “very heavy requirements on the dissemination of information and to assume away the costs of coalition formation.” Moreover, the inclusion into model of money as means of exchange, actually does not change a situation. Second, Edgeworth’s processes do not admit a revealed opportunity of production and consumption in nonequilibrium situations, now it is still open quest.\footnote{Hahn [9] notes the work of Hurwicz–Radner–Reiter [11], in which the process is considered in a stochastic context and, moreover, there is also shown that the production can be incorporated into model.} Third, the assumption, that a trade takes place only when the utilities of individuals are increasing, is not as harmless as it seems. In a true nonequilibrium world the individuals “trade even then there is no direct utility (or profit) gain from so doing because they wish to take advantage of arbitrage opportunities, speculating on their ability later to retrade at more advantage prices.” Moreover, Fisher writes: “Yet a crucial aim of stability theory must be to examine the question of whether arbitrage drives a competitive economy to equilibrium.”

On the other hand, Hahn’s process does not avoided shortcomings. One of them is that before purchasing something, one needs to sell something. Therefore many of the potentially interesting bargains may be not realized. In order to solve this problem, Arrow and Hahn [1] directly introduced money into model, using it as the intermediary-goods in any barter bargain, and imposed other additional assumptions. Mukherji [25] also criticizes Hahn’s process and specifies a main its shortcoming: the absence of revealed voluntary nature of barter bargains in process.

Really, if there is no specific model, explaining in an microeconomic way how and why (non-mutually beneficial) barter bargain is realized, voluntary can be understood only as a condition, attracting monotonous growth of utility of the individuals along trajectory, i.e., realizing bargain all its participants should win.

Finally, both processes are indirectly based on a hypothesis of auctioneer, since due to definitions they satisfy (5.5).

### 5.1.4 Edgeworth’s processes

We call so the processes of change of current resources allocation, described in continuous or discrete time, which are going without prices and, accordingly, there are no budget
constrains. Essentially the processes of this type are close to contractual processes without breaking of the contracts, see [9], p. 772–777, and also following section. The basic sense of process is, that the process generates a trajectory in space of allocations such that along trajectory there is the monotonous growth of individual utilities, and at least for one strictly, if a current point still is not Pareto optimum. A process of this type can be set, for example, via a rule of trade, described by functions \( g_i(\omega(t)), i \in \mathcal{I} \), similarly as it was made in the previous paragraph (however here there are no prices):

\[
\omega : [0, +\infty) \rightarrow \mathbb{R}^n, \\
\forall i \in \mathcal{I}, \quad \dot{\omega}_i(t) = g_i(\omega(t)) - \omega_i(t), \quad \sum_{i=1}^{n} \omega_i(t) = \sum_{i=1}^{n} \omega_i(0), \quad \forall t \in [0, +\infty).
\]

The functions \( g_i(\cdot) \) are continuous and satisfy to conditions (5.9), with the exception of budget constrains.\(^{23}\) It is simply to prove that every limiting point of such process is Pareto optimal.\(^{24}\)

There is a number of papers where Edgeworth’s processes are considered in stochastic context (see review [9], and also [11], [6]), where, in our terms, on the set of all mutually beneficial contracts some reasonable probability distribution is defined. The appropriate stochastic process converges to an Pareto optimum with probability 1 (we omit other specific features and assumptions).

There are also papers, in which Pareto boundary is attained by the efforts of coalitions of limited size (agents are not more than number commodities). The first result of this type was received in [28], see also [2], [7], [18], [8]. However, in so doing each active in barter process coalition carries out transition on intra-coalitional Pareto boundary (relative to current allocation) and all permissible coalitions are incorporated in a cycle, which is repeated infinite times (compare with (5.9)). Thus, essentially these processes are discrete in time.

There are also papers, in which the transition to core allocations are realized, see [8]. Here a stochastic context is also available, where the reaction of current blocking coalition replenishes with the reaction of supplementing coalition, that forms the transition from a current allocation in subsequent one. In such a way an allocation from core (if necessary the procedure is repeated infinitely) is attained with probability 1.

Easily to see, that in all of these directions is available contractual context and, moreover, the contract based language is simpler, more convenient and it can be better interpreted.

### 5.1.5 Strategic approach

This direction began to develop in the economic theory from the middle of 80’s of last century and was aimed to clarify the basic hypotheses of competitive equilibrium theory

\(^{23}\)This assumption is important for the process converges to Pareto optimum. Hahn, [9] p. 773, gives another description of process, requiring only a growth of utilities without restrictions on the derivative of process. Such process can be finished at an “irredundant” point, which is not Pareto optimal, an elementary example can be constructed: Let \( \alpha : [0, +\infty) \rightarrow [0, 1] \), supp \( \alpha = [0, \infty) \), \( \int_{0}^{\infty} \alpha(t)dt = 1 \) (one can take \( \alpha(t) = \frac{1}{(1+t)^2} \), \( \int_{0}^{1} \alpha(s)ds = 1 - \frac{1}{(1+t)} \)) and let \( v \) be a mutually beneficial contract such that \( u_i(x_i + \lambda v_i) \) strictly increases in \( \lambda \in [0, 1] \), \( \forall i \in \text{supp}(v) \), but \( x^* = x + v \) is not Pareto optimum. Then a trajectory of process \( z(t) = x + v \int_{0}^{t} \alpha(s)ds \) obeys all Hahn’s conditions and \( z(t) \rightarrow x + v \) for \( t \rightarrow +\infty \).

\(^{24}\)We can not give exact reference, but it seems that this (elementary) fact was clear to economists for a long time ago, and Uzawa is one of them.
in a context of a strategic game. An idea was to apply game theoretical methods to give the answers on such questions as: whence the prices are undertaken and who them defines, why the agents should accept the prices as given and why they cannot change them (a consumer is said to be a “price-taker”), that is equilibrium and perfect competition? The answers on these and other important theoretical questions are given in the analysis of some game in extensive form. These games belongs to a class of DMBG-games (dynamical matching and bargaining games), constructed by a model of economy in a special way. For lack of an opportunity to enter in detailed explanations, we specify only two sources of the literature [5], [17] and we describe only the basic idea of the approach in a context of one possible game model.

In economy there is a continuum of agents, presented by a finite number of types. Each type is characterized by initial endowments vector and by von Neumann-Morgenstern utility functions. Commodities are infinite divisible, time is discrete and is indexed by the natural numbers. At each time period each agent can meet the partner with some fixed probability. If some pair \( \{i, k\} \) of the agents has met, then, after the identification of type and current consumptions, \( x_i \) and \( x_k \), with equal probability one of them is selected, say \( i \), to propose a vector of goods \( z \in \mathbb{R}^l \) (makes the offer), to be transferred to him from his opponent \( k \). If the agent \( k \) accepts the offer, her/his consumption bundle changes on \( x_k - z \), and the partner’s one becomes \( x_i + z \) (only the allowable offers are under consideration, they are not to allow consumption bundles to leave the limits of consumption sets). If the agent does not accept the offer, the consumptions do not change. The individual, who does not accept the offer made to him, can leave the market at next time moment, no other individual (accepting the offer or not participating in the given round) can leave market. An agent, who never leaves market, receives utility equal to \(-\infty\) (thus, the consumption is possible only after leaving). A player’s strategy is a plan the prescribes her/his bargaining behavior in different trade situations for each period, depending on current consumption, type of partner, her/his current consumption and, the offer made by him (if so happen). The strategy of the agent depends on the realized (by him) earlier bargains and, if to him the offer is made, it may takes values: “accept the offer”, “reject and stay”, “reject and exit”.

It is supposed, that the agents of one type use common strategy. However, in view of the previous acts of trade, the different agents can have different current consumption bundles, but for each type only a finite number. Further, for constructed DMBG-game a concept market equilibrium is introduced, which is in fact a specialized kind of perfect Bayesian equilibrium and in discrete (equivalent) variant of game is presented as a sequential equilibrium. The basic result is the theorem, which states that in every market equilibrium each player leaves the market with probability 1, when her/his consumption bundle is equal to a bundle kept him in Walrasian equilibrium.

In spite of the fact that the methods of strategic approach are essentially different from contractual approach, we have found necessary to describe this direction, because it is aimed to solve similar problems of equilibrium theory, which are related with the validity and possibility to be correctly realize basic theoretical hypotheses. In our opinion, the most attractive part of strategic approach is the clear description of what and how the

\[25\] It is also called as a game in normal form.

\[26\] Apparently, it was Douglas Gale’s idea, but it seems that it is a suitable adaptation and development of ideas of previous researchers, see [5].

\[27\] In our terminology the offer \( z \) is an offer to sign the barter contract \((z, -z)\).
individuals make trades in uncertain market circumstances.

5.2 Examples of contractual processes: convergence and cycling

In this section we consider two additional examples revealing the character of contractual processes.\(^{28}\) The first one presents a computer realization of contractual process whereas the second example has an important theoretical meaning.\(^{29}\) It demonstrates that coalition contractual trajectory described in Section 1.3 may be cycled.

5.2.1 Convergence of UB-processes in an \(2 \times 3\) economy

In this item we describe a computer modelling of proper-contractual process for \(2 \times 3\) economy and its realization for a numerical example.

An economy with 2 agents and 3 goods is considered with Cobb–Douglas utilities in the logarithmic form:

\[
\begin{align*}
  u_1(x) &= a_{11} \ln(x_1) + a_{12} \ln(x_2) + a_{13} \ln(x_3), \quad x \gg 0, \\
  u_2(y) &= a_{21} \ln(y_1) + a_{22} \ln(y_2) + a_{23} \ln(y_3), \quad y \gg 0.
\end{align*}
\]

On the start of the program there are specified all parameters and an initial allocation. There are also determined specific parameters: \(\text{step} > 0\) (step) and \(\text{tochn} > 0\) (closeness). Contracts defining the movement of trajectory are found as a realization of a random value uniformly distributed on all mutual beneficial contracts with volume (the length of vector) not more than \(\text{step} > 0\).

Applying these data program finds an equilibrium and specifies a sequence of allocations \((x, y)^{(0)}, (x, y)^{(1)}, \ldots, (x, y)^{(n)}\) corresponding to the rules of proper-contractual process and represents the results graphically.

The economy parameters were varied and always convergency to equilibrium was observed. Table 1 reduces the results of the program work for the following data:

\[
\omega = (\omega_1, \omega_2) = ((9, 9, 2), (1, 5, 8)), \quad \text{step}=0.1, \quad \text{tochn}=1. \quad \text{The generated sequence of allocations} \quad (x, y)^{(n)} \quad \text{“arrives” to equilibrium}
\]

\[
(\bar{x}, \bar{y}) = ((1.761, 2.258, 9.454), (8.239, 11.742, 0.546)).
\]

Table 1 presents every 30th point of sequence \(x^{(1)}, x^{(2)}, \ldots, x^{(n)}\), i.e., only the consumption of 1st agent is described. The estimations of the contracts break are specified in fourth column and they are defined as a quotient of length of real progress of a trajectory to the length of signed contract, i.e., written \(\frac{\|x^{(n+1)}_j - x^{(n)}_j\|}{\|v^n\|}\) when the trajectory is moving by the maximal surface and it is applied “inside” if the current point is in the interior of area limited by maximal surface.

\(^{28}\)In [22] one can find a series of examples demonstrating possible convergency and non-convergency of non-benevolent of contractual processes.

\(^{29}\)These examples and programs were constructed in collaboration with Sergey Kolbin, IV year student of MMF NSU (during 2005–2006 years I supervised his bachelor diploma).
Table 1. Discrete proper contractual trajectory: computation results of each 30th step for 1st agent.

<table>
<thead>
<tr>
<th>1st good $x_1^{(k)}$</th>
<th>2nd good $x_2^{(k)}$</th>
<th>3rd good $x_3^{(k)}$</th>
<th>break degree $|(x,y)^{(k+1)}-(x,y)^{(k)}|$</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>9</td>
<td>2</td>
<td>$\omega_1$</td>
</tr>
<tr>
<td>9.002</td>
<td>8.956</td>
<td>2.048</td>
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<td>8.234</td>
<td>8.802</td>
<td>2.647</td>
<td>inside</td>
</tr>
<tr>
<td>7.455</td>
<td>8.353</td>
<td>3.469</td>
<td>inside</td>
</tr>
<tr>
<td>6.785</td>
<td>8.092</td>
<td>4.285</td>
<td>inside</td>
</tr>
<tr>
<td>6.044</td>
<td>7.681</td>
<td>5.148</td>
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</tr>
<tr>
<td>5.167</td>
<td>7.187</td>
<td>6.021</td>
<td>inside</td>
</tr>
<tr>
<td>4.638</td>
<td>6.673</td>
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</tr>
<tr>
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<td>6.347</td>
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</tr>
<tr>
<td>3.261</td>
<td>5.454</td>
<td>8.669</td>
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</tr>
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<td>0.943</td>
</tr>
<tr>
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<td>2.360</td>
<td>9.452</td>
<td>0.129</td>
</tr>
<tr>
<td>1.761</td>
<td>2.258</td>
<td>9.454</td>
<td>equilibrium</td>
</tr>
</tbody>
</table>

5.2.2 Recycling of coalition contractual CUB-process

Let us consider economy with 4 individuals which utilities define a family of indifference curves as it is presented in Figure 4. At the starting time moment ($t = 0$) the current allocation coincides with the initial endowments: $x_1(0) = \omega_1 = (9, 1)$, $x_2(0) = \omega_2 = (3, 3)$, $x_3(0) = \omega_3 = (3, 9)$, $x_4(0) = \omega_1 = (3, 4)$. Further, let in a time interval $(0, t_1)$ the coalition $\{1, 2\}$ be active, in an interval $(t_1, t_2)$ the coalition $\{2, 3\}$ is active, $\{3, 4\}$ is active in limits $(t_2, t_3)$ and in an interval $(t_3, t_4)$ the coalition $\{4, 1\}$ is active. Let’s assume also that in the nearest future the order of coalition activity is: $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}$ and that each time interval of activity is short enough. Suppose that for this interval of time a coalition have enough time only to break contracts signed in the past and have no time to sign a new contract.

Further let the signing of intra-coalition contracts be realized in the following way. On the segment $(0, t_1)$ a contract $v^{\{1,2\}} = (\omega_1 A_1, \omega_2 A_2)$ is signed and realized. The bundles of goods for the first and second individuals at the moment $t_1$ are $A_1$ and $A_2$, accordingly.
Figure 4: An example of $4 \times 2$ economy where CUB-process is cycling and does not converge to equilibrium.
Further, on \((t_1, t_2)\) a contract \(v^{(2,3)} = (A_2B_2, \omega_3A_3)\) is signed and realized by \(\{2, 3\}\). Thus, the bundles of goods for 2nd and 3rd individuals at the moment \(t_2\) are \(B_2\) and \(A_3\), accordingly. Further, on an interval \((t_2, t_3)\) a contract \(v^{(3,4)} = (A_3B_3, \omega_4A_4)\) is realized by \(\{3, 4\}\). So, the bundles of 3rd and 4th at the moment of time \(t_3\) are \(B_3\) and \(A_4\), accordingly. At last, on an interval \((t_3, t_4)\) a contract \(v^{(4,1)} = (A_4B_4, A_1B_1)\) is realized. In doing so the bundles of 4th and 1st at the moment \(t_4\) become equal to \(B_4\) and \(B_1\), accordingly. Thus, at the moment of time \(t_4\) the commodity bundle of agent \(1\) corresponds to the point \(B_1\) in the figure and the point \(A_1\) is the result of the first stage of barter exchange of the individual \(i\): for each individual it is realized on a time interval and with an appropriate partner. In each presented diagram there are specified the coalitions which signed contract resulting the specified consumer program. One can also see from the figures that all above-stated contracts are mutually beneficial. Moreover, completing these diagrams if necessary it is possible to design appropriate indifference curves in such manner that each signing of new contract is carried out in accordance with principles of proper-contractual process, where in each stage of barter exchange the role of “new” endowments plays the sum of initial endowments with a flow of goods received from contracts signed him in others coalitions.

When all acts of described bilateral contracts are realized, at the moment \(t_4\) the coalition \(\{1, 2\}\) is active again. However now 1st agent finds that it is favourable for him to partially break contract \(v^{(1,2)}\) with 2nd agent and he/she completely breaks it raising his/her utility from consumption. As a result new commodity bundles of 1st and 2nd agents are the vectors corresponding to the points \(C_1\) and \(C_2\) in the figure, accordingly. In so doing since the interval of coalition activity is short, the agents have no time to sign a new contract. Next active coalition is \(\{2, 3\}\). Now 2nd agent completely breaks the contract \(v^{(2,3)}\) signed him in the past in coalition \(\{2, 3\}\). Consumption bundles of 2nd and 3rd become \(\omega_2\) and \(C_3\), accordingly. Then coalition \(\{3, 4\}\) is next to be active. Now 3rd agent observes that it is favourable for him to break off the contract \(v^{(3,4)}\). As a result \(\omega_3\) and \(C_4\) become the consumption bundles of 3rd and 4th, accordingly. Finally coalition \(\{4, 1\}\) becomes an active one. However once again there is an agent, now it is 4th, which desires to break off coalition contract \(v^{(4,1)}\). The result of this is that \(\omega_4\) and \(\omega_1\) become 4th and 1st consumption bundles, accordingly.

Thus, one can see that the contractual trajectory has returned to the allocation of initial endowments and considered contractual process is cycled: the signing of last contract by coalition \(\{4, 1\}\) caused a breaking chain (likes avalanche) of all contracts! What can be said about this occasion? If the 1st individual were able to expect such development of events at a stage when he/she was signing contract with 4th agent or if at once after signing this contract he/she would limit appetites and has refused to break contract with 2nd agent, then the destruction of contractual structure of economy did not occur... However the behavior of such type should be clearly incorporated into the model of contractual process and this would mean essential modernization of our theoretical conceptions about proper-contractual processes.

5.3 Proofs

Proof of Lemma 3.1. So let alternative (i) be fulfilled and let starting at the moment \(\tau \geq 0\), only 1st individual can be active and second is passive for all \(t \geq \tau\). Further, let \(
\tilde{x}_1\) (1st agent bundle) be any limit point of trajectory. Define \(\tilde{x}_2 = \omega_1 + \omega_2 - \tilde{x}_1\) and show that allocation \((\tilde{x}_1, \tilde{x}_2)\) is Pareto optimal. Assuming contrary due to trade rule definition
we conclude $\langle \nabla u_1(\tilde{x}_1), v_1(\tilde{x}_1) \rangle > 0$ and by continuity this property has to be fulfilled in some neighborhood of the point $\tilde{x}_1$, i.e.

$$\exists \ \varepsilon > 0 : \ \langle \nabla u_1(x_1), v_1(x_1) \rangle > 0, \ \forall x_1 \in B_{2\varepsilon}(\tilde{x}_1),$$

where $B_{2\varepsilon}(\tilde{x}_1)$ is closed ball with the radius $2\varepsilon$ centered at the point $\tilde{x}_1$. Since the ball is a compact set and by continuity it is equivalent to

$$\exists \ \varepsilon > 0, \ \delta > 0 : \ \langle \nabla u_1(x_1), v_1(x_1) \rangle > \delta, \ \forall x_1 \in B_{2\varepsilon}(\tilde{x}_1). \quad (5.1)$$

We observe from this that one can come to a contradiction if one manages to show that the current point of trajectory is in the ball during infinite (by measure) time.

Really, since 2nd agent is passive (almost everywhere) and because 1st agent utility is increasing monotonically along the trajectory (starting at the moment $\tau$) the following estimations are fulfilled:

$$u_1(x_1(t)) - u_1(x_1(\tau)) = \int_{\tau}^{t} \frac{d u_1(x_1(\zeta))}{d \zeta} d \zeta = \int_{\tau}^{t} \langle \nabla u_1(x_1(\zeta)), \dot{x}_1(\zeta) \rangle d \zeta \geq$$

$$\geq \int_{\tau}^{t} \langle \nabla u_1(x_1(\zeta)), v_1(x_1(\zeta)) \rangle d \zeta \geq \int_{[\tau,t] \cap \Omega} \langle \nabla u_1(x_1(\zeta)), v_1(x_1(\zeta)) \rangle d \zeta \geq \delta \cdot \mu([\tau,t] \cap \Omega).$$

Here $\Omega \subset [\tau, +\infty]$ is the set of all time moments when a current point of trajectory $x_1(\zeta)$ is located in the ball $B_{2\varepsilon}(\tilde{x}_1)$ and $\mu([\tau,t] \cap \Omega)$ is Lebesgue measure of the set $[\tau,t] \cap \Omega$. If $\mu(\Omega) = +\infty$ we have $\mu([\tau,t] \cap \Omega) \to +\infty$ for $t \to +\infty$. Then due to the last estimation it has to be $u_1(x_1(t)) \to +\infty$ that is impossible since the set of all allocation is compact and utility function is continuous.

Let us show that $\mu(\Omega) = +\infty$. It is obvious if starting at some time moment $t \geq \tau$ all points of trajectory are located in the ball. In the contrary case one can find an enumerable set of moments $t_k, t'_k, k = 1, 2 \ldots$ such that $\|x_1(t_k) - \tilde{x}_1\| < \varepsilon$ and $t'_k > t_k$ is a closest after $t_k$ time moment when the trajectory leaves the ball, i.e.

$$\|x_1(t'_k) - \tilde{x}_1\| = 2\varepsilon \ \& \ \|x_1(\zeta) - \tilde{x}_1\| < 2\varepsilon, \ \forall \zeta \in [t_k, t'_k).$$

However in such a case we have an estimation:

$$\varepsilon \leq \|x_1(t'_k) - x_1(t_k)\| = \| \int_{t_k}^{t'_k} \dot{x}_1(\zeta) d \zeta \| \leq \int_{t_k}^{t'_k} \| \dot{x}_1(\zeta) \| d \zeta \leq c \int_{t_k}^{t'_k} d \zeta = c(t'_k - t_k),$$

where $c > 0$ is an upper bound for the norm of right hand part of the law (2.8), i.e. this is a value satisfying

$$c \geq \| \lambda_{\min}(x, v_1)(x_1 - \omega_1) + v_1(x_1) \|, \ \forall x_1 \in B_{2\varepsilon}(\tilde{x}_1).$$

Due to imposed assumptions and from the compactness and continuity of objects that we need it is easy to prove that the right hand part of this inequality is bounded from above and, therefore, such $c > 0$ does exist. As a result we have got the estimation

$$(t'_k - t_k) \geq \frac{\varepsilon}{c} > 0, \ \forall k = 1, 2, \ldots$$
Moreover via construction all intervals \([t_k, t_k']\) are pairwise non-intersected and \([t_k, t_k'] \subset \Omega, \forall k = 1, 2, \ldots\). Therefore, \(\mu(\Omega) = +\infty\). Thus we obtain a contradiction that proves Pareto optimality of the allocation under study.

To state the second part of lemma remember that every allocation from the interior of direct product of consumption sets which is Pareto optimal and simultaneously stable relative to the partial break of gross contract is an equilibrium, see Theorem 2.2 from [21]. □

**Proof of Lemma 3.2.** Let \(\tau'\) be some time moment when first agent is active. Now define a time moment when 1st agent is active \(\tau_1' \geq \tau'\) and such that it is earlier of *first* moment \(\tau'' > \tau'\) when 2nd agent is active and such that on the interval \((\tau_1', \tau'')\) both agents are passive. Here \(\tau_1'\) is the *latest* moment of 1st agent activity on the interval \([\tau', \tau'\)). Analogously, for 2nd agent one can find a moment \(\tau_2''\) as a moment of last his/her activity up to the nearest moment \(\tau''' > \tau''\) when 1st agent is active. In view of compactness and continuity of objects under study all considered time moments do exist. For example, \(\tau''\) and \(\tau_1'\) can be found by formulas

\[
\tau'' = \min\{t \in [\tau', +\infty) | \langle \nabla u_2(x_2(t)), x_2(t) - \omega_2 \rangle = 0\},
\]

\[
\tau_1' = \max\{t \in [\tau', \tau''] | \langle \nabla u_1(x_1(t)), x_1(t) - \omega_1 \rangle = 0\}.
\]

Further taking the point \(\tau'''\) as “initial” (i.e. instead of \(\tau'\)) in the described above procedure, one can find the moments \(\tau_2''\) and \(\tau_2''\), accordingly. Show that constructed in this way fragments of sequences that we need to be found obey the requirement (3.1). With this in mind first let us better understand the geometry of moving of a trajectory and reveal some peculiarities of this moving.

Really by construction on intervals \([\tau_1', \tau'']\) and \([\tau_2'', \tau_2''\]\) the 1st agent utility increases: it is so because only mutually beneficial contracts are signed during contractual process and also because in our intervals 2nd agent is passive. It has to be shown that for all points \(t\) from interval \([\tau'', \tau_1']\) the inequality \(u_1(x_1(t)) > u_1(x_1(\tau_1'))\) is fulfilled. Let us do it.

Now consider the moment \(\tau''\). By construction the following relations

\[
\langle \nabla u_1(x_1(\tau'')), v_1(x_1(\tau'')) \rangle > 0, \quad \langle \nabla u_2(x_2(\tau'')), v_1(x_1(\tau'')) \rangle < 0,
\]

\[
\langle \nabla u_2(x_2(\tau'')), x_1(\tau'') - \omega_1 \rangle = 0
\]

have to be true. Moreover if \(h_2(x_2(\tau'')) = \nabla u_2(x_2(\tau'')) - \nabla^2 u_2(x_2(\tau''))(x_1(\tau'') - \omega_1)\) satisfies \(^\text{30}\)

\[
\langle h_2(x_2(\tau'')), v_1(x_1(\tau'')) \rangle < 0,
\]

then the condition (2.5) of contracts break is violated and it means that a trajectory only “touched” with maximal surface at the point \(x_1(\tau'')\) and then “leaves” it. Therefore in a neighborhood of the moment \(\tau''\) a break of contracts does not occur and both utilities are locally increased. A break of contracts may occur only if

\[
\langle h_2(x_2(\tau'')), v_1(x_1(\tau'')) \rangle \geq 0
\]

\(^\text{30}\)Remember that \(x_2(\tau'') - \omega_2 = -(x_1(\tau'') - \omega_1)\) and \(v_2(x_2(\tau'')) = -v_1(x_1(\tau'')).\)
and if for small $\Delta t > 0$ at the points $\tau'' + \Delta t$ this inequality is strict. Thus after the “going through” the point $x_1(\tau'')$ a trajectory $x_1(t)$ will move some non-zero time in framework of $\varepsilon$-extension of a cone with the vertex at the point $x_1(\tau'')$ which is defined by inequalities:

$$\langle h_2(x_2(\tau'')), x_1(\tau'') \rangle \geq \langle h_2(x_2(\tau'')), x_1(\tau'') \rangle,$$

$$\langle \nabla u_2(x_2(\tau'')), x_1 \rangle \leq \langle \nabla u_2(x_2(\tau'')), x_1 \rangle = \langle \nabla u_2(x_2(\tau'')), \omega_1 \rangle.$$

More exactly, due to $\langle h_2(x_2(t)), \dot{x}_2(t) \rangle = 0$, see (2.2).

Further, on interval $[\tau_1'', \tau'']$ in the plane a trajectory $x_1(t)$ circumscribes an continuous curve with the ends $x_1(\tau_1'')$ and $x_1(\tau'')$ such that for $t \in (\tau_1'', \tau'')$ the points $x_1(t)$ are located strictly “below” than a point of maximal surface being intersected with the ray starting from $\omega_1$ and going through the point $x_1(t)$, because $\langle \nabla u_i(x_1(t)), x_i(t) - \omega_i \rangle > 0$, $i = 1, 2$. Moreover for $t < \tau''$ and close to $\tau''$ it has to be

$$\langle \nabla u_2(x_2(\tau'')), x_1(\tau'') \rangle < \langle \nabla u_2(x_2(\tau'')), x_1(t) \rangle \iff \langle \nabla u_2(x_2(\tau'')), \frac{x_1(t) - x_1(\tau'')}{\tau'' - t} \rangle > 0,$$

because $v_2(x_1(\tau'')) = -v_1(x_1(\tau'')) \approx \frac{x_1(t) - x_1(\tau'')}{\tau'' - t}$. Thus there exists a moment $t' < \tau''$ such that $x_1(t') - \omega_1 = \gamma(x_1(\tau'') - \omega_1)$ for some $0 < \gamma < 1$ and, simultaneously, all points of trajectory from interval $t \in [\tau_1'', t']$ obey $\langle \nabla u_2(x_2(\tau'')), x_1(t) \rangle < \langle \nabla u_2(x_2(\tau'')), \omega_1 \rangle$. The similar inequality has to be fulfilled for the points $x_1(t), t \in [\tau'', \tau_1]$ because contracts are mutually beneficial and 1st agent is passive on this time interval.

Further we are going to the final part of the proof. Assume that for some $t \in [\tau'', \tau_1]$ the inequality $u_1(x_1(t)) \leq u_1(x_1(\tau_1))$ is fulfilled. Now from the continuity and due to presented above reasonings it follows that there are moments $t'' < \tau''$ and $\tau'' < t''' \leq \tau_1''$ such that $x_1(t'') = x_1(t''')$ is true. Consider the first possible moment of this type (one needs to take minimal $t'''$ having this property). For $t'' > \tau_1''$ we have a contradiction since then our trajectory is cycling (due to the law of change is autonomous) and never arrives to a point on 1st agent maximal surface but it has to be so at the moment $\tau_1'' > t'''$. Therefore, it has to be $t''' = \tau_1''$. However $x_1(\tau_1'')$ is a point on 1st agent maximal surface where 2nd agent is passive. Hence there is a neighborhood of $x_1(\tau_1'')$ such that 1st agent utility strictly increases along every trajectory starting from any point from the neighborhood. Therefore for all small enough $\varepsilon > 0$ it has to be $u_1(x_1(t''' - \varepsilon)) < u_1(x_1(t''')$. Moreover for some $\varepsilon > 0$ no point $x_1(t), t \in (t''' - \varepsilon, t''')$ can be located on 2nd agent maximal surface (otherwise at the point $x_1(t''') = x(\tau_1'')$ both individuals are active that is possible only at an equilibrium which trajectory can never leave). Therefore the last moment of trajectory being on 2nd agent maximal surface, by definition this is the moment $\tau_1''$, has to be realized earlier the moment $t'''$ because the point $x_1(t''') = x(\tau_1'')$ is located on 1st agent maximal surface. Thus, it has to be $t''' > \tau_1''$ but this is impossible. The obtained contradictions finish the proof of Lemma 3.2.

Proof of Theorem 3.1. In conditions of the theorem the considered above alternatives (i), (ii) take place. So, it is enough to show, that for any alternative the contractual process converges to proper-contractual allocation.

Let alternative (i) be true. Let’s prove, that $x_1(t)$ converges to $\dot{x}_1$ when $t \to +\infty$. To do it let’s assume, that the trajectory has two different limit points $\ddot{x}_1 = (\ddot{x}_1^1, \ddot{x}_1^2)$, $\ddot{x}_2 = (\ddot{x}_2^1, \ddot{x}_2^2)$ and $\ddot{x}_1 \neq \ddot{x}_2$. Due to Lemma 3.1 both of them are Pareto optimal and
thus \( u_1(\tilde{x}_1) = u_1(\tilde{x}_1^2) \). Let’s assume, for example, that \( u_2(\tilde{x}_1) \leq u_2(\tilde{x}_2) \). Further consider any allocation represented as a convex combination of the given limit points with strictly positive coefficients, for example one can take \( \tilde{x}' = \frac{1}{2}\tilde{x}_1 + \frac{1}{2}\tilde{x}_2 \). Now, by virtue of strict concavity of utility functions conclude
\[
\quad u(\tilde{x}') \gg u(\tilde{x}_1) \iff u_1(\tilde{x}_1') > u_1(\tilde{x}_1), \quad u_2(\tilde{x}_2') > u_2(\tilde{x}_2),
\]
that contradicts Pareto optimality of allocation \( \tilde{x}_1 \). Thus all limit points of a trajectory coincide and, hence, the trajectory converges.

Further we analyze alternative (ii). With this purpose one can apply Lemma 3.2 and consider limit points of sequences \( \{x_1(\tau^1_k)\}_{k\in\mathbb{N}} \) and \( \{x_1(\tau^2_k)\}_{k\in\mathbb{N}} \). Without lost of generality one can think that these sequences are converged. Determine
\[
\tilde{x}_1^1 = \lim_{k\to\infty} x_1(\tau^1_k), \quad \tilde{x}_2^1 = \omega_1 + \omega_2 - \tilde{x}_1, \quad \tilde{x}_1^2 = \lim_{k\to\infty} x_1(\tau^2_k), \quad \tilde{x}_2^2 = \omega_1 + \omega_2 - \tilde{x}_2^1.
\]
In view of (3.1) we have
\[
u_1(\tilde{x}_1^1) = \sup_{t \geq t^1} u_1(x_1(t)) = u_1(\tilde{x}_1^2), \quad u_2(\tilde{x}_2^1) = \sup_{t \geq t^2} u_2(x_2(t)) = u_2(\tilde{x}_2^2).
\]
It is clear, that point \( \tilde{x}_1^1 \) is on the maximal surface of 1st agent and \( \tilde{x}_2^2 \) is on the maximal surface of 2nd, and that for both individuals the allocations are equivalent by utility. Further we shall show, that actually coincide not only utilities, but also allocations, i.e. \( \tilde{x}_1 = \tilde{x}_1^2 \). For two-goods economy this allocation will be obviously proper-contractual (equilibrium) since it is on the maximal surface of every agent.\(^{31}\)

Let’s assume now that \( \tilde{x}_1^1 \neq \tilde{x}_2^2 \). These points are on a common indifference curve of 1st agent, and, accordingly, the points \( \tilde{x}_1^2 \neq \tilde{x}_2^2 \) are on an indifference curve of 2nd individual. Reasoning in Edgeworth box, for example in coordinate system of 1st agent, we see that two points \( \tilde{x}_1^1 \neq \tilde{x}_2^2 \) are connected by two continuous curves which are the pieces of boundary of two (convex) sets of a utility level. Let’s connect the specified points by a linear segment, i.e., consider the set \( \{\gamma \tilde{x}_1^1 + (1 - \gamma)\tilde{x}_2^2 \mid 0 < \gamma < 1\} \). By virtue of strict concavity of utility functions, the value of utility at points of this segment is strictly more than utility level at its ends for both individuals. This implies that for one of agents the part of indifference curve, going through the points \( \tilde{x}_1^1, \tilde{x}_2^2 \) and placed strictly between these points, cannot intersect maximal surface of the agent, see Figure 5 (every ray going from initial endowments through one of considered points first intersects with one of two indifference curves and then it hits at a point of segment; therefore (via concavity) when point moves along the ray utility increases at the point of intersection with indifference curve). Let this be a case of 2nd agent indifference curve. Further for the 2nd agent indifference curve let us find a point \( x_1^{max} \) where 1st agent utility is maximal, i.e., define \( x_1^{max} \) from relation
\[
u_1(x_1^{max}) = \max\{u_1(x_1) \mid u_2(\omega_1 + \omega_2 - x_1) = u_2(\omega_1 + \omega_2 - \tilde{x}_1^1)\}.
\]
Obviously, that in Edgeworth box for 2nd agent indifference curve this point is placed strictly between points \( \tilde{x}_1^1, \tilde{x}_2^2 \). Further find neighborhoods \( V_1, V_2 \) of points \( \tilde{x}_1^1, \tilde{x}_2^2 \) and a neighborhood \( V_{max} \) of point \( x_1^{max} \) satisfying the following conditions:

\(^{31}\)It is not sufficient in general, but it will be so if the allocation is Pareto optimal.
Figure 5: “The impossible” moving of contractual trajectory along a limit path under alternative (ii) and $\tilde{x}_1 \neq \tilde{x}_2$.

1) At every point from $V_{\text{max}}$ the 1st agent utility is strictly more than his/her utility at any point from neighborhoods $V_1$, $V_2$;

2) For every point from $V_1$ if the trajectory passes through this point (i.e. it is starting from this point as a point of initial data in Cauchy problem) then it certainly passes through some point of neighborhood $V_{\text{max}}$.

Clearly that such neighborhoods can be found, since first it is possible to find neighborhoods satisfying 1) and then if necessary to reduce neighborhoods $V_1$, $V_2$. However now we come to the contradiction because 1st agent utility is non-monotonically changed in the part of trajectory where 2nd agent is passive — it contradicts to the property that every contract defined by trading rule is mutually beneficial.

\textbf{Proof of Lemma 4.1}. By definition (4.1) of the set $W^{fr}(x)$ one can equivalently to rewrite condition $W^{fr}(x) = \emptyset$ in the following way.

Let us define $B_i(x) = \{ z \in E \mid \langle \nabla u_i(x_i), z \rangle > 0 \}$ if the individual $i$ is passive and let $B_i(x) = \{ z \in E \mid \langle \nabla u_i(x_i), z \rangle > 0 \ & \langle h_i(x_i), z \rangle > 0 \}$ for the active individual. Then

$$\prod_i B_i(x) \cap L^c = \emptyset,$$

where, remember $L^c = \{(v_1, \ldots, v_n) \in E^I \mid \sum_i v_i = 0 \}$ is the space of contracts. Notice that $B_i(x) \neq \emptyset$ for active $i$ since $\nabla u_i(x_i) \neq 0$, $\langle \nabla u_i(x_i), x_i - \omega_i \rangle = 0$ and $\langle h_i(x_i), x_i - \omega_i \rangle < 0$ (hence vectors are non-collinear), and of course $B_i(x) \neq \emptyset$ for passive agents. Therefore one can apply separation theorem and find $\pi = (p_1, \ldots, p_n) \neq 0$ such that

$$\langle \pi, \prod_i B_i(x) \rangle \geq \langle \pi, L^c \rangle.$$
We see that functional $\pi$ is bounded from above on subspace $L^c$ which is possible only if $
abla u_i(x_i) = 0$ and, therefore, in standard manner one can conclude that $p_i = p_j$, $\forall i \neq j$. Denote $p = p_i \neq 0$. Further, it is easy to see that $\langle \pi, \prod_\mathcal{I} B_i(x) \rangle \geq 0$ is possible only if (applying $\pi = (p, \ldots, p)$)

$$\langle p, B_i(x) \rangle > 0, \quad \forall i \in \mathcal{I}$$

is true. Thus, for every active $i$ the inequality $\langle p, z \rangle > 0$ is a corollary of two inequalities: $\langle \nabla u_i(x_i), z \rangle > 0$, $\langle h_i(x_i), z \rangle > 0$, and for passive agent only of first of them. Now applying Farkas lemma (or again separation theorem) we conclude the existence of $\alpha_i \geq 0$, $\beta_i \geq 0$ demanded in the statement of lemma.

Proof of Lemma 4.2. It follows from the continuity of a trajectory $x(t)$ that for each individual $i$ the set of all moments $t > 0$ where he/she is passive is open on interval $(0, +\infty)$, because the set is defined via condition $\langle \nabla u_i(x_i(t)), x_i(t) - \omega_i \rangle > 0$. Due to Lemma 4.1 we have $p(x(t)) = \nabla u_i(x_i(t))$ in every moment $t$ where individual $i$ is passive. Therefore, since the gradient of utility continuously depends on trajectory points one can conclude that as the function of time $p(t)$ changes continuously in a neighborhood of $t$. Now we need to show that if $x(t) \neq 0$, i.e., if the point $x(t)$ is not equilibrium, and if a mutually beneficial exchange without break is impossible then a passive individual does exist. Assuming that all individuals are active via (4.3) (a mutually beneficial exchange without break is impossible) we have

$$p(t) = \alpha_i \nabla u_i(x_i(t)) + \beta_i h_i(x_i(t)), \quad \forall i \in \mathcal{I}.$$ 

Further let us multiply these equality on vectors $x_i(t) - \omega_i \neq 0$ and then sum the received equalities. As a result, since from the activity of the individuals we have $\langle \nabla u_i(x_i), x_i - \omega_i \rangle = 0, \forall i \in \mathcal{I}$ and due to $\sum (x_i(t) - \omega_i) = 0$ we obtain

$$0 = \sum \beta_i \langle h_i(x_i(t)), x_i(t) - \omega_i \rangle.$$ 

Since $\beta_i \geq 0$ and $\langle h_i(x_i), x_i - \omega_i \rangle < 0, \forall i \in \mathcal{I}$, then the last equality is possible only if $\beta_i = 0$ for all $i$, that is possible only in equilibrium. It is a contradiction.

Proof of Lemma 4.3. Fist let us choose $\varepsilon > 0$ so that if for $j \in \mathcal{I}$ inequality $\langle \nabla u_j(x_j(\tau)), x_j(\tau) - \omega_j \rangle > 0$ is fulfilled then for all $t \in (\tau, \tau + \varepsilon)$ the similar inequality $\langle \nabla u_j(x_j(t)), x_j(t) - \omega_j \rangle > 0$ is also fulfilled. It is possible in view of continuous dependence of a trajectory from time and since all functions participating in an inequality are continuous.

Further, let $\langle \nabla u_j(x_j(\tau)), x_j(\tau) - \omega_j \rangle = 0$ be fulfilled for some $j \in \mathcal{I}$, $j \neq i$, i.e., $j$ is another agent distinct from $i$ which is active at the moment $\tau$. Applying Definition 4.2 suppose, for example, that

$$\frac{\langle h_j(x_j(\tau)), v_i(x_i(\tau)) \rangle}{\langle h_i(x_i(\tau)), \omega_i - x_i(\tau) \rangle} < \frac{\langle h_j(x_j(\tau)), v_j(x_j(\tau)) \rangle}{\langle h_j(x_j(\tau)), \omega_j - x_j(\tau) \rangle}$$

holds. From a continuity of functions participating in the inequality it is possible also to find a neighborhood of point $x(\tau)$ in $\mathcal{A}(X)$ and a neighborhood of point $v(x(\tau))$ in the space of contracts $L^c$ such that the similar inequality is true for any point from these
neighborhoods replacing \( x(\tau) \) and \( v(x(\tau)) \), accordingly. Let \( \delta > 0 \) be such that

\[
\frac{\langle h_i(x_i), w_i \rangle}{\langle h_i(x_i), \omega_i - x_i \rangle} < \frac{\langle h_j(x_j), w_j \rangle}{\langle h_j(x_j), \omega_j - x_j \rangle}, \quad \forall x \in B_{\delta}(x(\tau)) \cap A(X), \forall w \in B_{\delta}(v(x(\tau))) \cap L^c
\]

is fulfilled, where \( B_{\delta}(y) \) denotes a ball of radius \( \delta > 0 \) centered at \( y \) in an appropriate space, and vectors \( h_i(x_i), h_j(x_j) \) are formally defined by formula (2.2) and are calculated at the designated point of space. Moreover, without lost of generality it is possible also to think that all contracts from \( B_{\delta}(v(x(\tau))) \cap L^c \) are beneficial at every point from \( B_{\delta}(x(\tau)) \cap A(X) \).

This obviously follows from the definition of the mutually beneficial contract and from the continuity of all functions participating in required inequalities. Besides it is possible to think that numerator and denominator in expressions from the last formula do not change a sign for all points of chosen neighborhoods and that this is true for any pair of active individuals at the moment \( \tau \). At last, reducing if necessary, \( \varepsilon > 0 \) can be chosen so that all points \( x(t) \) for \( t \in (\tau, \tau + \varepsilon) \) are in the limits of the chosen neighborhood \( B_{\delta}(x(\tau)) \) of \( x(\tau) \), i.e. for the time not more than \( \varepsilon > 0 \) the trajectory does not leave this neighborhood.

Further let us establish the validity of alternative (i). It is necessary to show that in conditions of (i) an arbitrarily chosen point of the trajectory \( x(t), t \in (\tau, \tau + \varepsilon) \) is located on the maximal surface of individual \( i \), i.e., that \( \langle \nabla u_i(x_i(t)), x_i(t) - \omega_i \rangle = 0 \) is fulfilled.

Assuming \( \langle \nabla u_i(x_i(t)), x_i(t) - \omega_i \rangle > 0 \) find a maximum of all those moments \( t' \in [\tau, t] \) where the current point of trajectory \( x(t') \) is located on the maximal surface of agent \( i \). As \( x(\tau) \) is on the maximal surface of the agent \( i \), this maximum does exist and obviously that at this moment the point of a trajectory is located on the maximal surface. Let \( s \) denote this maximum. Now we have \( \langle \nabla u_i(x_i(s)), x_i(s) - \omega_i \rangle = 0 \) and \( \langle \nabla u_i(x_i(\zeta)), x_i(\zeta) - \omega_i \rangle > 0 \) for all \( \zeta \in (s, t) \). By definition in the interval \((s, t)\) the law of change of a trajectory (2.8) is set by the contract \( v(x(\zeta)) \) and by (in general discontinuous) function \( \lambda^{\min}(\cdot) \) which in conditions of alternative (i) by the choice of \( \varepsilon \) and because individual \( i \) is passive for all \( \zeta \in (s, t) \) has to satisfy

\[
\lambda^{\min}(x(\zeta), v(x(\zeta))) > a \ni b > \frac{\langle h_i(x_i(\zeta)), v_i(x(\zeta)) \rangle}{\langle h_i(x_i(\zeta)), \omega_i - x_i(\zeta) \rangle} = g_i(x(\zeta), v(x(\zeta))) \tag{5.2}
\]

for some real \( a, b \). Further the vector \( h_i(x_i) \) participating the the right hand part of inequality (5.2) is defined by formula (2.2) and, therefore, it is the gradient of function \( F(x_i) = \langle \nabla u_i(x_i), x_i - \omega_i \rangle \) which defines maximal surface by equation \( F(x_i) = 0 \). So we have \( F(x_i(s)) = 0 \) and the value \( F(x_i(t)) \) can be found by formula

\[
F(x_i(t)) = \int_s^t \frac{F(x_i(\zeta))}{d\zeta} d\zeta = \int_s^t \langle \nabla x_i, F(x_i(\zeta)), \dot{x}_i(\zeta) \rangle d\zeta,
\]

where the function under integral is summarized (since \( x(\cdot) \) is an absolute continuous function). Substituting expression of under-integral functions (\( \dot{x}_i(\zeta) \) via the law of trajectory) in view of (5.2) and \( \langle h_i(x_i(\zeta)), x_i(\zeta) - \omega_i \rangle < 0, \forall \zeta \in (s, t) \) we obtain the following estimation

\[
F(x_i(t)) = \int_s^t \langle h_i(x_i(\zeta)), \lambda^{\min}(\zeta)(x_i(\zeta) - \omega_i) + v_i(x_i(\zeta)) \rangle d\zeta \leq
\]

\[
a \int_s^t \langle h_i(x_i(\zeta)), (x_i(\zeta) - \omega_i) \rangle d\zeta + \int_s^t \langle h_i(x_i(\zeta)), v_i(x_i(\zeta)) \rangle d\zeta \leq
\]

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\[(a-b) \int_s^t \langle h_i(x_i(\zeta)), x_i(\zeta) - \omega_i \rangle d\zeta + \int_s^t \langle h_i(x_i(\zeta)), g_i(\zeta)(x_i(\zeta) - \omega_i) + v_i(x_i(\zeta)) \rangle d\zeta =
\]
\[(a-b) \int_s^t \langle h_i(x_i(\zeta)), x_i(\zeta) - \omega_i \rangle d\zeta.
\]

The last equality in the chain of estimations is true because the second integral (summand) is equal to zero: by definition of \(g_i(\zeta) = g_i(x(\zeta), v(x(\zeta)))\) in (5.2) and due to
\[
\langle h_i, \frac{\langle h_i, v_i \rangle}{\langle h_i, \omega_i - x_i \rangle}(x_i - \omega_i) + v_i \rangle = 0, \quad x_i \neq \omega_i.
\]

Since \(\int_s^t \langle h_i(x_i(\zeta)), x_i(\zeta) - \omega_i \rangle d\zeta < 0\) then as a result we conclude
\[
F(x_i(t)) = \langle \nabla u_i(x_i(t)), x_i(t) - \omega_i \rangle < 0,
\]
that contradicts the initial assumption. Thus alternative (i) has proven.

In a part of the proof of alternatives (ii) and (iii) we only note that it can be done in accordance with the same method as stated above. The difference consists in the formulation of an requirement similar to (5.2) but written down concerning other parameters: only this thing is important to obtain the key estimations. For example, for the proof of alternative (ii), for the individual \(j \neq i\) which is active at the moment \(\tau\) for a suitable time interval one needs to apply
\[
\lambda^{\text{min}}(x(\zeta), v(x(\zeta))) < a < b = \frac{\langle h_j(x_j(\zeta)), v_j(x_j(\zeta)) \rangle}{\langle h_j(x_j(\zeta)), \omega_j - x_j(\zeta) \rangle} = g_j(x(\zeta)), v(x(\zeta)).
\]

Lemma 4.3 has proven. \(\blacksquare\)

Proof of Theorem 4.1. Let us determine a time moment \(\tau > 0\) as a first moment when the mutually beneficial exchange without partial break of gross contract \(x(t) - \omega\) is impossible. If there are no such moments then alternative (i) is realized. However, if the set of all such moments is not empty then being the closed set it always has the minimal element. Therefore moment \(\tau\) is determined correctly. Further, if \(\dot{x}(\tau) = 0\) then \(x(\tau)\) is an equilibrium and, therefore, the alternative (ii) is realized. Let \(\dot{x}(\tau) \neq 0\). Now we are able to apply Lemma 4.1 and can conclude that at the moment \(\tau\) there is at least one really active individual. Really, otherwise \(\beta_i(\tau) = 0\) for all \(i \in \mathcal{I}\) that is possible only in equilibrium. At the same time really passive individuals also do exist that in our conditions (non-equilibrium allocation where beneficial exchange without break is impossible) can be proved similarly as it was done in Lemma 4.2. So, let \(\mathcal{I}^{\text{ra}}(\tau) \subset \mathcal{I}\) be a nonempty set of all really active individuals at the moment \(\tau\) such that \(\mathcal{I}^{\text{ra}}(\tau) \neq \mathcal{I}\).

Further one can apply the fact that our trajectory is non-degenerate and show that alternative (i) of Lemma 4.3 is true. We need to prove that \(\lambda_i(x(\tau), v(x(\tau))) < 0\) for some active individual \(i \in \mathcal{I}\) at the moment \(\tau\). To do it we need to show \(\langle h_i(x_i(\tau)), v_i(x_i(\tau)) \rangle < 0\). With this in mind one can apply Lemma 4.1 and conclude the existence of a vector \(p(\tau) \neq 0\) and, for each \(i\), numbers \(\alpha_i(\tau) \geq 0\), \(\beta_i(\tau) \geq 0\) such that (4.3) is carried out:
\[
p(\tau) = \alpha_i(\tau) \nabla u_i(x_i(\tau)) + \beta_i(\tau) h_i(x_i(\tau)), \quad \forall i \in \mathcal{I}.
\]
Further, for each really active individual \(i\) from \(\mathcal{I}^{\text{ra}}(\tau) \neq \emptyset\) multiply the appropriate equality on vector \(u_i(x_i(\tau))\) and then sum the received equalities. The obtained result can be written down as

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\[
\langle p(\tau), \sum_{T^a(\tau)} v_i(x_i(\tau)) \rangle - \sum_{T^a(\tau)} \alpha_i(\nabla u_i(x_i(\tau)), v_i(x_i(\tau))) = \\
\sum_{T^a(\tau)} \beta_i(\tau) \langle h_i(x_i(\tau)), v_i(x_i(\tau)) \rangle.
\]

Since each summand in the right hand part of this equality has a positive factor (strictly more than zero) we shall receive required property if it will be established that the value in the left hand part of equality is negative. But it is so because by the definition of contract \(\sum_{T^a(\tau)} v_i(x_i(\tau)) = -\sum_{T \setminus T^a(\tau)} v_i(x_i(\tau))\) which is mutually beneficial for really passive individuals gives
\[
\langle p(\tau), \sum_{T^a(\tau)} v_i(x_i(\tau)) \rangle = -\langle p(\tau), \sum_{T \setminus T^a(\tau)} v_i(x_i(\tau)) \rangle < 0.
\]

Therefore, the left hand part of previous equality is the summation of negative and non-positive values and as a whole it is negative.

So, at present moment we have proven that alternative (i) of Lemma 4.3 is realized. This implies that for some \(\varepsilon > 0\) on the interval \((\tau, \tau + \varepsilon)\) the contractual process goes with a break of gross contract and only one agent is active. Let \(i_0\) be this individual. Only this individual (from complementarity slackness conditions from Lemma 4.1) can be really active on the interval \((\tau, \tau + \varepsilon)\) and, therefore,
\[
\beta_{i_0}(t) > 0, \quad \beta_j(t) = 0, \quad \forall j \neq i_0, \quad \forall t \in (\tau, \tau + \varepsilon).
\]

At last, applying Lemma 4.2 we can in these relations pass to limit for \(t \to \tau + 0\) (all functions are continuous) concluding that \(\beta_{i_0}(\tau) > 0\) and \(\beta_j(\tau) = 0, \quad \forall j \neq i_0\). Thus, the individual \(i_0\) is sole really active agent on the interval \([\tau, \tau + \varepsilon)\).

Below, on former assuming that \(x(\tau)\) is not equilibrium we prove the validity of alternative (iii).

With this in mind we first show, that for every \(t > \tau\) the mutually beneficial exchange without partial break of gross contract \(x(t) - \omega\) is impossible. Assuming opposite, find \(t' > \tau\) as infimum of all moments where the exchange without break is possible. It is clear, that the set of all such moments forms an open set on \((\tau, +\infty)\) and \(t'\) can not belong to it. Therefore, at the moment \(t'\) the mutually beneficial exchange without break is impossible. Besides \(x(t')\) can not be an equilibrium since the exchange goes after moment \(t'\). Now we can apply Lemmas 4.1, 4.3 and reasoning similar to described above we can conclude the existence of \(\delta > 0\) such that at every point of interval \([t', t' + \delta)\) the contractual process is realized with partial break of contracts that contradicts to the choice of moment \(t'\).

Further we define \(\tau_1 = \tau\) and find the moment \(\tau_2\) as infimum of all those moments of time from \((\tau_1, +\infty)\) when there is at least one another active individual distinct from \(i_0\). If there are no such moments then \(\tau_2 = +\infty\) and everything is proven. Let us assume \(\tau_2 < +\infty\) and show that \(x(\tau_2)\) is an equilibrium. First of all note that at the moment \(\tau_2\) only individual \(i_0\) can be really active. Really at this moment the mutually beneficial exchange without break is impossible and on the interval \((\tau_1, \tau_2)\) only \(i_0\) is active, therefore, applying Lemma 4.1 we obtain
\[
\beta_{i_0}(t) > 0, \quad \beta_j(t) = 0, \quad \forall j \neq i_0, \quad \forall t \in (\tau_1, \tau_2).
\]
Further, in view of Lemma 4.2 all functions $\beta_i(\cdot)$, $i \in I$ are continuous on $[\tau_1, \tau_2]$ and, passing to limits by $t \to \tau_2 - 0$ we conclude

$$\beta_{i_0}(\tau_2) \geq 0, \quad \beta_j(\tau_2) = 0, \quad \forall j \neq i_0.$$ 

Further, if $\beta_{i_0}(\tau_2) \neq 0$ we are in conditions of alternative (i) from Lemma 4.3 and hence for some $\varepsilon > 0$ on interval $(\tau_2, \tau_2 + \varepsilon)$ does exist only one active individual. By the choice $\tau_2$ this individual can not be $i_0$. Therefore $i_0$ is passive on $(\tau_2, \tau_2 + \varepsilon)$ and once again via Lemma 4.1 we conclude $\beta_{i_0}(t) = 0$ on the interval $(\tau_2, \tau_2 + \varepsilon)$. However $\beta_{i_0}(t)$ is continuous function on $(\tau_2, \tau_2 + \varepsilon)$ by Lemma 4.2. Now passing to a limit by $t \to \tau_2 + 0$ we conclude $\beta_{i_0}(\tau_2) = 0$. The received contradiction proves that $\beta_{i_0}(\tau_2) = 0$. However it was established above that $\beta_j(\tau_2) = 0, \forall j \neq i_0$. This is possible only for equilibrium point (since at the point $x(\tau_2)$ gradients of all individuals are pairwise collinear this is Pareto optimum which is also stable relative to the partial break of gross contract). Theorem 4.1 has proven.

Proof of Theorem 4.2. Theorem 4.1 and its Corollary 4.1 can be applied in the conditions of this theorem. Thus, each limit point of a trajectory is an equilibrium. Further we show, that in conditions of Theorem 4.2 every benevolent trajectory can have only one limit point.

Assume contrary and let $x, y$ be two different limit points of a trajectory. Let’s consider a linear segment with the ends $x, y$, i.e., the set $\{\gamma x + (1 - \gamma)y \mid 0 \leq \gamma \leq 1\}$. Across each point $z(\gamma) = \gamma x + (1 - \gamma)y$, $\gamma \in (0, 1)$ of the segment one can conduct a hyperplane so that points $x, y$ are strictly in the different half-spaces. For example, such hyperplane $H(\gamma)$ can be conducted as a hyperplane which has $y - x$ as a vector of its normal. It is clear, that in such manner we can define a family of pairwise-not-crossed hyperplanes depending on parameter $\gamma \in (0, 1)$ such that our two different limit points of trajectory are placed in two different open half-spaces defined by $H(\gamma)$. Hence, when time elapses the trajectory crosses every hyperplane infinite number of times and any limit point of these points of crossing is also a limit point of trajectory and, therefore, this is an equilibrium. Thus, the economy has a continuum of different equilibria, since for different $\gamma$ we have limit points from different parallel hyperplanes. However each regular economy has a finite number of equilibria. This contradiction proves that there is the only limit point and, hence, benevolent UB-contractual process converges to an equilibrium.
List of notations and special symbols

Let $C$ be a subset of topological vector space, then:

- $\text{int} C$ denotes the interior of $C$,
- $\text{co} C$ denotes convex hull,
- $\text{cl} C$ or $\overline{C}$ denotes the closure of $C$.

$E = \mathbb{R}^l$ is $l$-dimension commodity space;

$\mathbb{R}^l_+ = \{x \in \mathbb{R}^l \mid x \geq 0\}$ is the positive cone of commodity space (positive orthant);

$\mathbb{R}^l_{++} = \text{int} \mathbb{R}^l_+$ is the cone of all strictly positive vectors;

$I = \{1, \ldots, n\}$ is the set of agents;

$L = E^n = E^I$ is the space of economy allocations;

$L^e = \{(v_1, \ldots, v_n) \in E^I \mid \sum_I v_i = 0\}$ is the space of contracts;

$X_i = E^+_i = \mathbb{R}^l_+$ is consumption set of agent $i \in I$;

$X = \prod_{i \in I} X_i$;

$\omega_i \in X_i$ is the vector of $i$th agent initial endowments;

$\omega = (\omega_i)_{i \in I} \in X$ is the vector of initial endowments of all traders of the economy;

$u_i : X_i \to \mathbb{R}$ is $i$’s agent utility function;

$\mathcal{A}(X) = \{x = (x_i)_{i \in I} \in X \mid \sum_{i \in I} x_i = \sum_{i \in I} \omega_i\}$ is the set of all feasible allocations;

$\{x_i \in X_i \mid \langle \nabla u_i(x_i), x_i - \omega_i \rangle = 0\}$ is the maximal surface of agent $i \in I$;

$h_i(x_i) = \nabla u_i(x_i) + \nabla^2 u_i(x_i)(x_i - \omega_i)$ is the normal vector for tangent hyperplane to $i$’s maximal surface at the point $x_i \in X_i$;

$\mathcal{I}^a(x) = \{i \in I \mid \langle \nabla u_i(x_i), x_i - \omega_i \rangle = 0\}$ is the set of all active individuals at $x \in \mathcal{A}(X)$;

$W^{fr}(x) = \{w \in L^e \mid \langle \nabla u_i(x_i), w_i \rangle > 0, \forall i \in I \& \langle h_i, w_i \rangle > 0, \forall i \in \mathcal{I}^a(x)\}$ is the set (possible empty) of all mutually beneficial contracts that being signed do not attract the break of aggregated contract $x - \omega$.

References


